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ABSTRACT

Anon-linear parabolic system is derived to describe incompressible nuclear waste disposal contamination in porous media. Galerkin method is applied for the pressure equation. For the concentration, a kind of partial upwind finite element scheme is constructed. The finite element solution satisfies discrete maximum principle and converges to the solution in norm $L^2(0,T,L^2(\Omega))$. 
1. Introduction

The proposed disposal of high-level nuclear waste in underground repositories is an important environmental topic for many countries. Decisions on the feasibility and safety of the various sites and disposal methods will be based, in part, on numerical models for describing the flow of contaminated brines and groundwater through porous or fractured media under severe thermal regimes caused by the radioactive contaminants.

A fully discrete formulation is given in some detail to present key ideas that are essential in code development. The non-linear couplings between the unknowns are important in modeling the correct physics properties of flow.

In this model, we obtain a convection-diffusion equations which represent a mathematical model for a case of diffusion phenomena in which underlying flow is present; $\Delta w$ and $b\nabla w$ correspond to the transport of $w$ through the diffusion process and the convection effects, respectively, where $\nabla$ and $\Delta$ denoted respectively the gradient operator and the Laplacian operator in the spatial coordinates.

In this paper we will consider the fluid flow in porous media using a Galerkin method for the pressure equation and a kind of partial upwind finite element scheme is constructed for the convection dominated saturation (or concentration). For more details of this subject see Douglas, 2002; Douglas, 2001 and Huang, 2000.

2. Model Equations

The model for incompressible flow and transport of contaminated brine in porous media can be described by a differential system that can be put into the following form, (see Douglas, 2002).
Fluid:
\[
\begin{align*}
(a) & \quad \nabla \cdot u = -q + R_s \\
(b) & \quad u = -\frac{k(x)}{\mu(c)} \nabla P = -a(c) \nabla P
\end{align*}
\] (1)

Brine:
\[
\phi \frac{\partial c}{\partial t} + u \cdot \nabla c - \nabla . (E_c \nabla c) = g(c)
\] (2)

Heat:
\[
d \frac{\partial T}{\partial t} + c_p u \cdot \nabla T - \nabla . (E_h \nabla T) = Q(u, T, c, p)
\] (3)

Radionuclide:
\[
\phi K_i \frac{\partial c_i}{\partial t} + u \cdot \nabla c_i - \nabla . (E_c \nabla c_i) = f_i(c, c_1, \ldots, c_N)
\] (4)

With the boundary conditions
\[
\begin{align*}
(a) & \quad u \cdot n = 0 \quad \text{on} \quad \Gamma \\
(b) & \quad (E_c \nabla c - cu) \cdot n = 0 \quad \text{on} \quad \Gamma \\
(c) & \quad (E_c \nabla c_i - c_i u) \cdot n = 0 \quad \text{on} \quad \Gamma \\
(d) & \quad (E_h \nabla T - c_p T u) \cdot n = 0 \quad \text{on} \quad \Gamma \\
(e) & \quad \frac{\partial p}{\partial n} = 0 \quad ; \quad (x, t) \in \Gamma \times (0, T)
\end{align*}
\] (5)
And the initial conditions

\[
\begin{align*}
(a) \quad p(x,0) &= p_0(x) \quad ; \quad x \in \Omega \\
(b) \quad c(x,0) &= c_0(x) \quad ; \quad x \in \Omega \\
(c) \quad c_i(x,0) &= c_{i_0}(x) \quad ; \quad x \in \Omega \\
(d) \quad T(x,0) &= T_0 \quad ; \quad x \in \Omega
\end{align*}
\]

(6)

Where \( n \) is the unit outer normal to \( \Gamma \), \( x \in \Omega \subset R^2, t \in (0, T] \), \( u \) is the Darcy velocity, \( P \) the pressure, \( \phi = \phi c_w \), \( q = q(x, t) \) is a production term,

\[
R^j_s = R^j_s(c) = [c_s \phi K_s f_s / (1 + c_s)](1 - c)
\]

is a salt dissolution term, \( k(x) \) is the permeability of the rock, and \( \mu(c) \) is the viscosity of the fluid, is dependent upon \( c \), the concentration of the brine in the fluid, \( T \) is the temperature of the fluid, \( d_s = \phi c_p + (1 - \phi) \rho R c_p \),

\[
\tilde{E}_i = D c_{pw} + K_m I, K_m = k_m / \rho_0, D = (D_j)
\]

\[
= (\alpha_r |u| \delta_r + (\alpha_L - \alpha_r) u_i u_j / |u|),
\]

and

\[
Q(u, T, c, p) = -\{[\nabla U_0 - c_p \nabla T_0] u + [U_0 + c_p (T - T_0)] + (p / \rho)][-q + R^j_s]
\]

\[-q_L - q_H - q_u.
\]

\[
E_c = D + D_m I, \quad \text{and} \quad g(c) = -c \{[c_s \phi K_s f_s / (1 + c_s)](1 - c) - q c_i + R^j_s
\]

\[c_i \text{ is the trace concentration of the } i\text{-th radionuclide, and}
\]

\[
f_i(c, c_1, c_2, ..., c_N) = c_i \{q - [c_j \phi K_j f_j / (1 + c_j)](1 - c)\} - q c_i + q_{ci}
\]

\[+ \sum_{j=1}^N k_j \lambda_j K_j \phi c_j - \lambda_i K_i \phi c_i.
\]

The reservoir \( \Omega \) will be taken to be of unit thickness and will be identified with a bounded domain in \( R^2 \). We shall omit
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gravitational terms for simplicity of exposition, no significant mathematical questions arises the lower order terms are included. We define

\[ L_2(\Omega) = \{ v \mid v \text{ is defined on } \Omega \text{ and } \int_{\Omega} v^2 \, dx < \infty \} . \]

\[ H^1(\Omega) = \{ v \in L_2(\Omega) : \frac{\partial v}{\partial x_i} \in L_2(\Omega), i = 1, \ldots, n \} . \]

and introduce the corresponding scalar products and norms

\[ (v, w) = \int_{\Omega} vw \, dx , \quad \|v\|_{L_2(\Omega)} = \left( \int_{\Omega} v^2 \, dx \right)^{1/2} , \]

\[ (v, w)_{H^1(\Omega)} = [v w + \nabla v \cdot \nabla w] dx , \quad \|v\|_{H^1(\Omega)} = \left( \int_{\Omega} \left[ v^2 + |\nabla v|^2 \right] \, dx \right)^{1/2} . \]

We also define \( H^1_0(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma \} \), where \( \Gamma \) is the boundary of \( \Omega \) and we equip \( H^1_0(\Omega) \) with the same scalar product and norm as \( H^1(\Omega) \).

\[ H^2(\Omega) = \{ v \in L_2(\Omega) : \frac{\partial v}{\partial x_i}, \frac{\partial^2 v}{\partial x_i^2} \in L_2(\Omega), i = 1, \ldots, n \} . \]

With norm \( \|v\|_{H^2(\Omega)} = \left( \int_{\Omega} \left[ v^2 + |\nabla v|^2 + |\Delta v|^2 \right] \, dx \right)^{1/2} , \)

Let \( W^s_x(\Omega) \) be the Sobolev space on \( \Omega \) with norm

\[ \|v\|_{W^s_x(\Omega)} = \left( \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha v}{\partial x^\alpha} \right\|^2_{L^2(\Omega)} \right)^{1/2} , \]

with the usual modification for \( s = \infty \). When \( s = 2 \), let \( \|\cdot\|_{W^2_x} = \|\cdot\|_{H^2} = \|\cdot\|_{k} \).

(see Johnson, 1987).

We assume that

(A1)

\[ a(c), R'(c), g(c), f_i(c, c_1, \ldots, c_n), Q(u, Tc, p) \in C^1_0(R) \]

\[ \phi(x), K, d_2 \in H^1(\Omega), q \in L^\infty(0, T; H^1(\Omega)) \]
The solutions of the problem (1-6) are regular:
\[ c(x,t) \in L^2(0,T;H^2(\Omega)) \]
\[ P(x,t) \in L^\infty(0,T;H^{r+1}(\Omega)), \quad (r \geq 2) \]
\[ c_1, c_m \in L^\infty(0,T;H^1(\Omega)) ; p_1, p_m \in L^\infty(0,T;L^r(\Omega)) \]

For any \( \phi \in L^2(\Omega) \), the boundary values problem:
\[ \frac{\partial \phi}{\partial n} = 0 \quad , \quad x \in \Gamma \]
There exists unique solution \( \phi \in H^2(\Omega) \) and a positive constant \( M \) such that \( \|\phi\|_2 \leq M \|\phi\| \).

3. Finite Element Spaces
Consider a regular family \( \{T_h\} \) of triangulation defined over \( \Omega \), where \( h \) is the longest diameter of a triangular element with the triangular \( T_h \), we have a set of close triangles \( \{e_i\} \) \((1 \leq i \leq N_e)\) and a set of nodes \( \{P_i\} \) \((1 \leq i \leq N_p + M_p)\) where \( P_i \) \((1 \leq i \leq N_p)\) are interior nodes in \( \Omega \) and \( P_j \) \((N_p+1 \leq j \leq N_p + M_p)\) are boundary nodes on \( \Gamma \). We put \( h_s \) to be the maximum side length of triangles and \( k \) to be minimum perpendicular length of triangles for all \( e \in T_h \).

**Definition 3.1:** A family \( T_h \) of triangulations is of weakly acute type, if there exists a constant \( \theta_0 > 0 \) independent of \( h \) such that, the internal angle \( \theta \) of any triangle \( e_i \in T_h \) satisfies \( \theta_0 \leq \theta \leq \pi/2 \).

Let \( \phi_i(p), (1 \leq i \leq M) \), be the continuous function in \( \Omega \) s.t. \( \phi_i(p) \), is linear on each \( e \in T_h \) and \( \phi_i(p_j) = \delta_{ij} \) for any nodal point \( p_j \).
We denote $M_h$ as the linear span of $\phi_i, (1 \leq i \leq M)$, i.e., a finite dimensional subspace of $H^1(\Omega)$

$M_h = \{ z_h \mid z_h \in C(\Omega); z_h \text{ is a linear function on } e, \forall e \in T_h \}$.

And a subspace of $H^1_0(\Omega)$

$M_{oh} = \{ z_h \mid z_h \in M_h; z_h(P_k) = 0, k = M + 1, \ldots, K \}$.

We associate the index set

$\Delta_i = \{ j \neq i : P_j \text{ is adjacent to } P_i \}$. Let $P_i, P_j, P_k$, be three vertices of triangular element $e$ and $\lambda_i, \lambda_j, \lambda_k$ be barycentric coordinates.

We have the following definitions see (Hu & Tian(1992))

**Definition 3.2**: with each vertex $P_i$ belonging to triangle $e$, the barycentric subdivision $\Omega_i^e$ is given by:

$$\Omega_i^e = \{ P \mid P \in e \text{ ; } \lambda_i(P) \geq \lambda_j(P), \lambda_i(P) \geq \lambda_k(P), \forall P_j \in e \}$$

and the barycentric domain $\Omega_i$ associated with vertex $P_i$ in $\Omega$ is given by $\Omega_i = \cup \Omega_i^e, e \in T_h$.

**Definition 3.3**: with the characteristic function $\mu_i(x)$ of barycentric domain $\Omega_i$, the mass lumping operator $\hat{\cdot} : w \in C(\Omega) \to \hat{w} \in L_\infty(\Omega)$ is defined by

$$\hat{w}(p) = \sum_{i} w(p_i) \mu_i(p)$$

Using interpolation theory in Sobolev space (see Ciarlet,1978) and inverse inequality, with step length $h_c$ we have the relation between $\hat{w}$ and $w$ from the following lemma

**Lemma 3.1**: There exists a constant $C$ such that:

$$\| w - \hat{w} \|_{\omega, p} \leq Mh_c \| w \|_{\omega, p}, \quad \forall w \in M_h, p \geq 1 \quad (7)$$

$$\| w_h \|_1 \leq Mh_c^{-1} \| w_h \|, \quad \forall w_h \in M_{oh} \quad (8)$$
**Proof:** (see Hu and Tian, 1992)

**Lemma 3.2:** There exists constants \( C_1, C_2 > 0 \) such that:
\[
C_1 \| w \|_{0,p} \leq \| \hat{w} \|_{0,p} \leq C_2 \| w \|_{0,p}, \quad \forall w \in M_h
\]  
(9).

**Proof:** (see Hu and Tian, 1992)

**Definition 3.4:** Let \( \{ M_h \} \) be a family of finite dimensional subspaces of \( C(\Omega) \), which is a piecewise polynomial space of degree less or equal to \( r \) with step length \( h_P \) and the following property: for \( P \in [1,\infty] \), \( r \geq 2 \), there exists a constant \( M \) such that for \( 0 \leq q \leq 2 \) and \( \phi \in w^{r+1}_p(\Omega) \):
\[
\inf_{x \in \{ M_h \}} \| \phi - x \|_{q,p} \leq M h^{r+1-q} \| \phi \|_{r+1,p}
\]

Similarly, we define \( \{ N_h \} \) as a family of finite-dimensional subspace of \( C(\Omega) \times C(\Omega) \), which is piecewise polynomial space of degree less or equal to \( r-1 \) with the similar property as \( M_h \) and \( 0 \leq q \leq r-1 \).

We also assume that the families \( \{ M_h \} \) and \( \{ N_h \} \) satisfy inverse inequalities:
\[
\| \phi \|_{L^2} \leq M h^{-1}_p \| \phi \|, \quad \| \nabla \phi \|_{L^2} \leq M h^{-1}_p \| \nabla \phi \|, \quad \forall \phi \in M_h
\]
(see Manaa, 2000).

**4. Error Estimates**

Let \( \tau > 0 \) is a time step and \( N_\tau = \frac{T}{\tau} \). We use a Galerkin finite element method for the pressure and velocity and partial upwind finite element scheme for brine, radionuclides, and heat equation.

Let \( C^0 \in M_h \) be a \( L^2(\Omega) \)-projection of \( c^0 \) in \( M_h \):
\[
(C^0 - c^0, z_h) = 0 \quad \forall z_h \in M_h.
\]
We can get \( P^0 \in V_h \) such that
\[
\int_{\Omega} P^0 dx = 0, \quad \left( \frac{k(x)}{\mu(C^0)} \nabla P^0, \nabla v \right) = (-q^0, v) + (R^0, v), \quad \forall v \in V_h \text{ and } P^0 \in W_h \text{ from }
\]
\[
U^0 = - \frac{k(x)}{\mu(C^0)} \nabla P^0 = - a(C^0) \nabla P^0
\]
If the approximate solution
\[
\{ P^m, U^m, C^m, C^m_i (i = 1, 2, \ldots, N), T^m \} \in V_h \times W_h \times M_h \times M_h^N \times R_h
\]
is known, we want to find
\[
\{ P^{m+1}, U^{m+1}, C^{m+1}, C^{m+1}_i (i = 1, 2, \ldots, N), T^{m+1} \} \in V_h \times W_h \times M_h \times M_h^N \times R_h
\]
at \( t = t^{m+1} \), with three steps. Let \((.,.)\) denote the inner product in \( L^2(\Omega)\)

**Step 1.** Find \( C^{m+1} \) for \( m = 0, 1, \ldots, N_t - 1 \), such that
\[
(D^m z^m_i, z^m_h) + (E^m \nabla C^{m+1/2}, \nabla z^m_i) + R(U^m, C^{m+1/2}, C^{m+1/2}_i, z^m_h) = (g(C^{m+1/2}, z^m_h) \forall z^m_h \in M_h (10)
\]
where \( D^m C^m = (C^{m+1} - C^m) / \tau \), \( C^{m+1/2} = (C^{m+1} + C^m) / 2 \) and
\[
R(U^m, C^{m+1/2}, z^m_h) = \sum_{i=1}^{M} z^m_i \sum_{j=\alpha}^{\beta} \beta^m_{ij} (\alpha^m_{ij} C^{m+1/2}_i + \alpha^m_{ji} C^{m+1/2}_j)
\]
with \( z^m_i = z^m_h(P_i) \), \( C^{m+1/2} = C^{m+1/2}(P_i) \), and \( \beta^m_{ij} = \int_{\Gamma_j} U^m n_{ij} d\Gamma \), here \( n_{ij} \)
is the unit outer normal to \( \Gamma_{ij} \). The partial upwind coefficients should be required that (see hu & Tian(1992))

\[
(a) \quad \alpha^m_{ij} + \alpha^m_{ji} = 1
\]
\[
(b) \quad \max\{1 / 2, 1 - \rho_{ij}^{-1}\} \leq \alpha_{ij} \leq 1, \text{ if } \beta_{ij} \geq 0, \quad \max\{1 / 2, 1 - \rho_{ij}^{-1}\} \leq \alpha_{ji} \leq 1, \text{ if } \beta_{ij} < 0
\]

**Step 2 -Find \( P^{m+1} \) such that:
\[
\int_{\Omega} P^{m+1} dx = 0, \quad \left( \frac{k(x)}{\mu(C^{m+1})} \nabla P^{m+1}, \nabla v \right) = (-q^{m+1}, v) + (R^0, v), \quad \forall v \in V_h \quad \text{(13)}
\]
Step3 - Find $U^{m+1}$ as:

$$U^{m+1} = -k(x)\mu(C^{m+1})\nabla P^{m+1} = -a(C^{m+1})\nabla P^{m+1}$$  \(14\)

**Lemma (4.1):** Let $\overline{p} \in V_h$ be the elliptic projection of $p \in H^1(\Omega)$ into $V_h$ defined by $(a(c)\nabla \overline{p}, \nabla v) = (a(c)\nabla p, \nabla v)$, $\forall v \in V_h$ then there exists a constant $k_1$ such that $\|\overline{p} - p\| + h_p\|\nabla \overline{p} - \nabla p\| \leq k_1\|p\|_{l_1}, h_r^{m+1}$

**Proof:** (see Quarteroni, 1997).

**Some Important Remarks:**
1. $U^{m+1}.n = 0$ in $\Gamma$ \(15\)
2. if $\|U^l - u^l\| \leq Mh_p$ \(0 \leq l \leq m\) then $\|U^{m+1} - u^{m+1}\| \leq Mh_p$.
   and if $\|U^m - u^m\| \leq Mh_p$, then $\|\nabla U^m\| \leq M_i$ \(16\)
   see (Manaa, 2000).
3. We will make the inductive assumption that if $\|U^l\|_{l_\infty} \leq k^* \ (0 \leq l \leq m)$, then $\|U^{m+1}\|_{l_\infty} \leq k^*$ \(17\)
4. From (Quarteroni, 1997) if $T_h$ is regular triangulation of weakly acute type we have $\|w_h\|_{l_1} \leq \sqrt{6/k}\|\hat{w}_h\|$ \(18\)

**Theorem (4.1):** C satisfies discrete mass conservation law

$$\int_{\Omega} \phi D_{\hat{C}^m} d\Omega = \int_{\Omega} \hat{g}^{m+1/2}(C) d\Omega \ , \ m = 1,2,\ldots,N_c$$  \(19\)

**Proof:** In (10), let $z_h = 1$, then $(Ec\nabla C^{m+1/2}, \nabla 1) = 0$ and
\[ R(U^m, C^{m+1/2}, l) = \sum_{i=1}^{M} \sum_{j \in \mathcal{A}_i} \beta_{ij}^m C_{ij}^{m+1/2} = \] 
\[ \sum_{e \in T_h} \sum_{p, p' \in e, i \in j} (\beta_{ij}^m C_{ij}^{m+1/2} - \beta_{ij}^m C_{ij}^{m+1/2}) = 0 \]

Then (19) holds.

**Lemma (4.2):** Let 
\[ \bar{c} : [0, t] \to M_h \] such that 
\[ (Ec \nabla c - \bar{c}, \nabla z) - \lambda(c - \bar{c}, z) = 0 \quad \forall z \in M_h, \]

\[ t \in J \] and let 
\[ c - \bar{c} = \xi \]

\[ \|\xi\| \leq M h_c, \quad \left\| \frac{\partial \xi}{\partial t} \right\| \leq M h_c^2, \quad \left\| \frac{\partial^2 \xi}{\partial t^2} \right\| \leq M h_c \]

**Proof:** see (Ciarlet, 1978).

**Lemma (4.3):** For all \( z_h \in M_h \) and \( \xi = c - \bar{c}, \ \bar{\xi} = C - \bar{C}, \)

\[ \left\| (u^m \nabla c^{m+1/2}, z_h) - R(U^m, C^{m+1/2}, z_h) \right\| \leq M(h_c + \xi_{m+1/2}^2 + \| z_h \|^2 + \varepsilon \| \nabla z_h \|^2) \] 

\[ \left\| \nabla \xi_{m+1/2}^2 + \| U^m - u^m \|^2 + \| z_h \|^2 + \varepsilon \| \nabla z_h \|^2 \right\| \]

where \( \varepsilon > 0 \) is arbitrary small constant.

**Proof:**
\[ ((u^m \nabla c^{m+1/2}, z_h) - R(U^m, C^{m+1/2}, z_h) = ((u^m - U^m) \nabla c^{m+1/2}, z_h) \]
\[ + (U^m \nabla (c^{m+1/2} - C^{m+1/2}), z_h) + (U^m \nabla C^{m+1/2}, z_h - \bar{z}_h) \]
\[ + [(U^m \nabla C^{m+1/2}, \bar{z}_h) - R(U^m, C^{m+1/2}, z_h)] = J1 + J2 + J3 + J4 \]

With (A1) and (A2) we have:
\[ J1 = ((u^m - U^m) \nabla c^{m+1/2}, z_h) \leq M \| u^m - U^m \| \| z_h \| \]
\[ \leq M (\| u^m - U^m \|^2 + \| z_h \|^2) \]
Using (A1), (17), (18) and (9) we have:
\[ J_2 = (U^m \cdot \nabla (c^{m+1/2} - c^{m-1/2}), \hat{z}_h) \leq M \| \nabla (c^{m+1/2} - c^{m-1/2}) \| \| \hat{z}_h \|
\]
\[ \leq M \| \nabla \varsigma^{m+1/2} \| + \| \nabla \varsigma^{m+1/2} \| \| \hat{z}_h \|
\]
\[ \leq M \left( \frac{\sqrt{6}}{k} \| \varsigma^{m+1/2} \|^2 + \frac{\sqrt{6}}{k} \| \varsigma^{m+1/2} \|^2 \right) + M \| \hat{z}_h \|^2
\]
\[ \leq M \| \varsigma^{m+1/2} \|^2 + \| \varsigma^{m+1/2} \|^2 + \| \hat{z}_h \|^2 )
\]

From (A2), (7), (8), (17) and (18) we have
\[ J_3 = (U^m \nabla C^{m+1/2}, z_h - \hat{z}_h) \leq M \| U^m \| \| \nabla C^{m+1/2} \| \| z_h - \hat{z}_h \|
\]
\[ \leq M \| \nabla \varsigma^{m+1/2} \| \| z_h - \hat{z}_h \| + M \| \nabla \varsigma^{m+1/2} \| \| z_h - \hat{z}_h \|
\]
\[ + M \| \nabla \varsigma^{m+1/2} \| \| z_h - \hat{z}_h \| = k_1 + k_2 + k_3
\]
\[ k_1 \leq M \left( h_c \| \varsigma^{m+1/2} \|^2 \right) + \| \nabla z_h \|^2
\]
\[ k_2 \leq M \left( h_c \| \varsigma^{m+1/2} \|^2 + \| z_h \|^2 \right)
\]
\[ k_3 \leq M \left( h_c \| \varsigma^{m+1/2} \| \right)
\]
\[ J_3 \leq M \left( h_c \| \varsigma^{m+1/2} \| \right) + \| \nabla \varsigma^{m+1/2} \| + \| \varsigma^{m+1/2} \|^2 + \| z_h \|^2 + \varepsilon \| \nabla z_h \|^2
\]

Using (9), (11), (16) and (17) implies
\[ J_4 \leq \sum_{i=1}^{M} \sum_{f \in N G_y} \int (C^{m+1/2} - C_y^{m+1/2}) U^m n_y d \Gamma + \| C^{m+1/2} \| \nabla U^m \| \hat{z}_h \|
\]

Since \( C_y^{m+1/2} = \alpha_y^m C_i^{m+1/2} + \alpha_j^m C_j^{m+1/2} \) so
\[ J_4 \leq \sum_{x \in T_h} \sum_{y \in T_y} \sum_{P, P \in e} (z_i - z_j) \int (C^{m+1/2} - C_y^{m+1/2}) U^m n_y d \Gamma +
\]
\[ M \left( C^{m+1/2} - \bar{c}^{m+1/2} + \bar{c}^{m+1/2} - c^{m+1/2} + c^{m+1/2} \right) \| \hat{z}_h \|
\]

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\begin{align*}
\leq M \sum_{e \in T_h} h_e |\nabla(z_h)| \sum_{P_j \neq P_j} \int_{\Gamma} C^{m+1/2} - C^{m+1/2}_{ij} |d\Gamma
+ M \left( \|z_h\| + \|z_h\| + \|z_h\| \right)
\end{align*}

And in element e we have
\[ |C^{m+1/2} - C^{m+1/2}_{ij}| \leq 2|\nabla(C^{m+1/2} |e|)h_e\]

Then
\[ J_4 \leq 3M 2 \sum_{e \in T_h} h_e^2 \|\nabla(z_h)\| \|\nabla(C^{m+1/2} |e|)\]
\[ + M \left( \|z_h\| + \|z_h\| + \|z_h\| \right) \]
\[ \leq M (h_e^2 + \|z_h\|^2 + \|z_h\|^2 + \|z_h\|^2 + \|z_h\|^2 ) + \varepsilon \|\nabla z_h\|^2 \]

Hence \( J_1 + J_2 + J_3 + J_4 \) Implies (20).

**Lemma (4.4):** There exists a positive constant \( k_2 \) such that:
\[ \|\nabla P^{m+1} - \nabla \bar{P}^m \| \leq k_2 \|C^{m+1} - \bar{C}^{m+1}\| \]  \( (21) \)

**Proof:**

We have that:
\begin{align*}
(a(C^{m+1})\nabla P^{m+1}, \nabla v) &= (q^{m+1}, v) + (R_y(C^{m+1}), v), \quad (22) \\
(a(c^{m+1})\nabla \bar{P}^m, \nabla v) &= (q^{m+1}, v) + (\bar{R}_y(c^{m+1}), v), \quad (23)
\end{align*}

Subtracting (23) from (22), we get
\begin{align*}
(a(C^{m+1})\nabla (P^{m+1} - \bar{P}^m), \nabla v) &= ([a(C^{m+1}) - a(c^{m+1})]\nabla \bar{P}^m, \nabla v) \\
&= (\bar{R}_y(C^{m+1}) - (\bar{R}_y(c^{m+1}), v)
\end{align*}

Let \( v = P^{m+1} - \bar{P}^m + V_h \), then:
\[ \|\nabla (P^{m+1} - \bar{P}^m)\|^2 \leq \left| (a(C^{m+1})\nabla (P^{m+1} - \bar{P}^m), \nabla (P^{m+1} - \bar{P}^m)) \right| \]
\[ = \left| ([a(c^{m+1}) - a(C^{m+1})]\nabla \bar{P}^m, \nabla (P^{m+1} - \bar{P}^m)) + (\bar{R}_y(C^{m+1}) - (\bar{R}_y(c^{m+1}), P^{m+1} - \bar{P}^m)) \right| \]

Using (A1) we have
\[
\leq M \left\| \nabla \bar{P}^{m+1} \right\| + \left\| C^{m+1} - c^{m+1} \right\| + \left\| \nabla (P^{m+1} - \bar{P}^{m+1}) \right\|
\]

\[
+ \left\| C^{m+1} - c^{m+1} \right\| + \left\| P^{m+1} - \bar{P}^{m+1} \right\|
\]

with lemma (4.1) and (A2) if \( h_P > 0 \) is sufficiently small

\[
\left\| \nabla \bar{P}^{m+1} \right\| \leq \left\| \nabla P^{m+1} \right\| + k_i \left\| P^{m+1} \right\| h_P < M , \text{ so we have (21).}
\]

**Lemma (4.5):** There exists a positive constant \( k_3 \) such that

\[
\left\| U^{m+1} - u^{m+1} \right\| = k_3 \left( \left\| C^{m+1} - c^{m+1} \right\| + h_P \right) \quad (24)
\]

**Proof:**

\[
\left\| U^{m+1} - u^{m+1} \right\| = a(C^{m+1}) \nabla P^{m+1} - a(c^{m+1}) \nabla P^{m+1} \right\|
\]

\[
\leq a(C^{m+1}) \nabla (P^{m+1} - p^{m+1}) + a(C^{m+1}) - a(c^{m+1}) \left\| \nabla P^{m+1} \right\|
\]

From (A1) and (A2) we have

\[
\leq \text{const.} \left\| \nabla P^{m+1} - \nabla p^{m+1} \right\| + \left\| C^{m+1} - c^{m+1} \right\| \left\| \nabla p^{m+1} \right\|
\]

we have

\[
\left\| \nabla P^{m+1} - \nabla p^{m+1} \right\| \leq \left\| \nabla P^{m+1} - \nabla p^{m+1} \right\| + \left\| \nabla p^{m+1} \right\|
\]

using lemma (4.2) and lemma (4.4), we get

\[
\left\| \nabla P^{m+1} - \nabla p^{m+1} \right\| \leq k_2 \left\| C^{m+1} - c^{m+1} \right\| + k_1 \left\| p^{m+1} \right\| h_P , \text{ from (A2), (24) holds.}
\]

**Theorem (4.2):** For all \( m \leq l \leq N \), if \( \tau \leq \tau_0 \), then

\[
\left\| C^{l+1} - c^{l+1} \right\| \leq M (\tau + h_c + h_P) , \text{ where } M \text{ independent of } \tau \text{ and } h_c .
\]

**Proof:** Multiply eq.(2) by \( z_h \) and integrating by parts we obtain for

\[ t = (m+1/2)\tau . \text{ Let } w^{m+1/2} = w_c, (m+1/2)\tau \text{ and } w^{m+1/2} = (w^{m+1} + w^m) / 2, \text{ then}
\]

\[
(\phi D_c^{m}, z_h) + (Ec^{m+1/2}, z_h) + (u^m \nabla c^{m+1/2}, z_h) =
\]

\[
(g(c^{m+1/2}, z_h) + (\phi(D_c^{m}, \frac{\partial c}{\partial t}^{m+1/2}, z_h) +
\]

\[
+ (Ec^{m+1/2} - Ec^{m+1/2}, z_h) + (u^m \nabla c^{m+1/2} - \bar{u}^{m+1/2} \nabla c^{m+1/2}, z_h) \quad (25)
\]

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Let \( e = c - C = (c - \bar{c}) - (C - \bar{c}) = \zeta - \bar{\zeta} \), and subtract (10) from (25), we obtain:

\[
(\hat{\phi}D_c \hat{c}^m, \hat{z}_h) + (Ec \nabla c^{m+1/2}, \nabla z_h) = (R(U^m, C^{m+1/2}, z_h) - (u^{m+1/2} \nabla c^{m+1/2}, z_h) + ((g(c^{m+1/2}), z_h)) + (\phi(D_c \hat{c}^m - \frac{\partial \phi}{\partial t}|_{m+1/2}), z_h)
\]

\[
+((E \nabla c^{m+1/2} - E \nabla c^{m+1/2}), \nabla z_h) + ((u^{m} \nabla c^{m+1/2} - u_{m+1/2} \nabla c^{m+1/2}), z_h)
\]

Hence

\[
(\hat{\phi}D_c \hat{c}^m, \hat{z}_h) + (Ec \nabla c^{m+1/2}, \nabla z_h) = (\hat{\phi}D_c \hat{c}^m, \hat{z}_h) + (Ec \nabla c^{m+1/2}, \nabla z_h) +

((u^{m+1/2} \nabla c^{m+1/2}, z_h) - R(U^m, C^{m+1/2}, z_h)) + ((\hat{g}(C^{m+1/2}), \hat{z}_h)) - (g(c^{m+1/2}), z_h)) +

((\hat{\phi}D_c \hat{c}^m, \hat{z}_h) - (\hat{\phi}D_c \hat{c}^m, \hat{z}_h)) - (\hat{\phi}(D_c \hat{c}^m - \frac{\partial \phi}{\partial t}|_{m+1/2}, z_h)
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6
\]

In (26), let \( z_h = \hat{\zeta}^{m+1/2} \in M_h \), and using (A1) the left-hand side is

\[
\geq \frac{\phi}{2\tau} \left| \nabla \hat{\zeta}^{m+1/2} \right|^2 - \frac{1}{\tau} \left| \nabla \hat{\zeta}^m \right|^2 + c_0 \left| \nabla \hat{\zeta}^{m+1/2} \right|^2
\]

From (A1), we have:

\[
I_1 = (\hat{\phi}D_c \hat{c}^m, \hat{z}_h) \leq M \left( \|D_c \hat{c}^m\|^2 + \|\nabla \hat{\zeta}^{m+1/2}\|^2 \right)
\]

Using (7) and (8), we have

\[
\|D_c \hat{\zeta}^m\| \leq \|D_c \hat{\zeta}^m\| + \|D_c \hat{\zeta}^m - D_c \hat{\zeta}^m\| \leq \|D_c \hat{\zeta}^m\| + Mh_c \left| \nabla \hat{\zeta}^m \right|
\]

From (9) we have

\[
\left| \nabla \hat{\zeta}^{m+1/2} \right| \leq M \left( \|\nabla \hat{\zeta}^{m+1/2}\| + \|\nabla \hat{\zeta}^m\| \right)
\]

\[
I_1 \leq M \left( \|\nabla \hat{\zeta}^{m+1/2}\|^2 + \|\nabla \hat{\zeta}^m\|^2 + \|D_c \hat{\zeta}^m\|^2 + h_c^2 \left| \nabla \hat{\zeta}^m \right|^2 \right)
\]

Using (A1), we have:

\[
I_2 \leq M \left( \|\nabla \hat{\zeta}^{m+1/2}\|^2 + \epsilon \left| \nabla \hat{\zeta}^{m+1/2} \right|^2 \right)
\]
From lemma (4.3)
\[ I_3 \leq M (h_c^2 + \| \xi_{m+1/2} \|^2 + \| \xi_{m+1/2} \|^2 + \| U_m - u^m \|^2 ) + \varepsilon \| \nabla \xi_{m+1/2} \|^2 \]

let \( \theta = \phi \frac{c^{m+1} - c^m}{\tau} = \phi \left( \frac{\partial c}{\partial t} \bigg|_{m+1/2} + 1/24 \frac{\partial^3 c}{\partial t^3} \right)^2 \), using (A1),(A2) and (Raviort,1979) we get
\[ I_4 \leq M h_c (1 + \tau^2) |z_h| + M h_c (1 + \tau^2) |z_h| \leq M h_c \| z_h \| + M h_c \| z_h \| + M h_c \| z_h \| + M h_c \| z_h \| \leq M (h_c^2 + h_c^2 \tau^4) + \varepsilon \| \nabla \xi_{m+1/2} \|^2 \]
\[ I_5 \leq M (h_c^2 + \| \xi_{m+1/2} \|^2 + \| \xi_{m+1/2} \|^2 ) + \varepsilon \| \nabla \xi_{m+1/2} \|^2 \]

Let \( I_6 = K_1 + K_2 + K_3 \), so
\[ K_1 \leq M (\tau^2 + \| \xi_{m+1/2} \|^2 + \| \xi_{m} \|^2 ), \quad K_2 \leq M \tau^2 + \varepsilon \| \xi_{m+1/2} \|^2, \quad K_3 \leq M (\tau^2 + \| \xi_{m+1/2} \|^2 + \| \xi_{m} \|^2 ) \]

Then we get
\[ I_6 \leq M (\tau^2 + \| \xi_{m+1} \|^2 + \| \xi_{m} \|^2 ) + \varepsilon \| \nabla \xi_{m+1/2} \|^2 \]

Equation (26) can be written now
\[ \frac{\partial}{\partial t} \| \xi_{m+1} \|^2 - \| \xi_{m} \|^2 \leq M (\| \xi_{m+1} \|^2 + \| \xi_{m} \|^2 ) + \frac{2c}{c_0} \| \nabla \xi_{m+1/2} \|^2 \leq M \| \xi_{m+1} \|^2 + \| \xi_{m} \|^2 \]
\[ + \| D_1 \xi_{m+1} \|^2 + h_c^2 \| D_2 \xi_{m} \|^2 + \| \nabla \xi_{m+1/2} \|^2 + \| \nabla \xi_{m+1/2} \|^2 + \| \xi_{m+1/2} \|^2 + \| U_m - u^m \|^2 + h_c^2 + \tau^2 + \varepsilon \| \nabla \xi_{m+1/2} \|^2 \]

Take the summation from 0 to \( l \), where \( m \leq l \leq \mathcal{N}_e \), and \( c_0 = \bar{c}^0 \) so \( \xi_0 = 0 \)
\[ \frac{\partial}{\partial t} \| \xi_{m+1} \|^2 \leq M \| \xi_{m+1} \|^2 \tau + M_2 \| D_2 \xi \|^2 L_2^2 (L^2) + M_2 h_c^2 D_3 \xi \| L_{h^2} (\psi_h) \|^2 \]
\[ + M_2 \sum_{m=0}^{1} \| \xi |L^2 | \tau + M_2 (h_c^2 + \tau^2) + M_3 \sum_{m=0}^{1} \| U_m - u^m \|^2 \tau \]

Where \( \| w \|^2_{L_{\mathcal{K}}(\tau)} = \sum_{m=0}^{N_e} \| w^m \|^2 \tau \), using (7) and (A1), we get
\[ \|s^{i+1}\| \leq \|s^i\| + 2\|M_{c}\| \|s^{i+1}\| \leq \|s^i\| + 2\|M_{c}\| \sqrt{\frac{6}{k}} \|s^{i+1}\| \leq M \|s^{i+1}\| \]

\[ \phi \|s^{i+1}\| \geq M_{0} \|s^{i+1}\|, \quad \text{So} \]

\[ \|s^{i+1}\|^2 \leq M_{c} \sum_{m=0}^{i+1} \|s^m\|^2 \|s^{i+1}\| + M_{2}^{\frac{1}{2}} \|D_{\xi}\|^2 + \|D_{\xi}\|^2 + \|\bar{D}_{\xi}\|^2 \]

From lemma (4.6) we have

\[ \sum_{m=0}^{i+1} \|U^m - u^m\|^2 \leq M_{c} \sum_{m=0}^{i+1} \|s^m\|^2 \|s^{i+1}\| + M_{2} \|s^{i+1}\|^2 + \|s^{i+1}\|^2 + h_{p}^{2r} \] (28)

From (27) and (28), we have

\[ \|s^{i+1}\|^2 \leq M_{i} \sum_{m=0}^{i+1} \|s^m\|^2 \|s^{i+1}\| + M_{2} \|s^{i+1}\|^2 + h_{p}^{2r} \]

From Gronwall inequality, we get:

\[ \|s^{i+1}\| \leq M (h_{c} + \|s^{i+1}\|) \]

it is \( \|C^{i+1} - c^{i+1}\| \leq M (h_{c} + \|s^{i+1}\|) \)

**Theorem 4.3** with the assumption (A1) ~ (A3), if \( h_{c} = O(h_{p}), \tau = O(h_{p}) \),

Then \( \|C - c\|_{L^{i+2}(i^{2})} + \|U - u\|_{L^{i+2}(i^{2})} \leq M (\|s^{i+1}\| + h_{c} + h_{p}^{2r}) \)

**Proof:**

We will prove first the inductive assumption (17). We have

\[ \|U^{m+1}\|_{L_{\omega}} \leq \|a(C^{m})\|_{L_{\omega}} \|\nabla P^{m+1}\|_{L_{\omega}} \leq k_{s} \|\nabla P^{m+1}\|_{L_{\omega}} \]

But

\[ \|\nabla P^{m+1}\|_{L_{\omega}} \leq \|\nabla P^{m+1} - \bar{P}^{m+1}\|_{L_{\omega}} + \|\bar{P}^{m+1}\|_{L_{\omega}} \]

\[ \leq Mh_{p}^{-1} \|\nabla P^{m+1} - \bar{P}^{m+1}\| + k_{s} \leq 2k_{s} \]

When \( h_{c} = o(h_{p}), \tau = o(h_{p}) \) \( (r \geq 2), \|\nabla P^{m+1}\|_{L_{\omega}} \leq k \),
\[
\left\| U^{m+1} - u^{m+1} \right\| \leq k_j \left( \|C^{m+1} - c^{m+1}\| + h' \right), \quad \forall m = 0,1,\ldots,N_z \quad (29)
\]

And from theorem (4.2)

\[
\left\| C^{m+1} - c^{m+1} \right\| \leq M (h + \tau + h') \ , \quad \forall m = 0,1,\ldots,N_z \quad (30)
\]

Equations (29) and (30) implies

\[
\left\| C^{m+1} - c^{m+1} \right\| + \left\| U^{m+1} - u^{m+1} \right\| \leq M \left( \tau + h' + h \right), \quad \forall m = 0,1,\ldots,N_z
\]

Which is complete the proof.
REFERENCES


