An Alternative Computational Algorithm to Study the Sensitivity Analysis of SPP

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Abstract

In this paper, we present a new computational algorithm which can be used to study the sensitivity analysis of GPP and SPP of degree of difficulty greater than zero. To study the sensitivity analysis of GPP (Dinkal & Kohgenberger [2]), one has to use a subroutine to solve a system of linear equation to get the values of optimal primal variables. Our proposed algorithm is thus more effective in that no new subroutines (such as one for computing the inverse matrices j_{ij}^{-1} are needed). The generalized algorithm which was presented by Mohan & Al-Bayati to study the sensitivity analysis of SPP [1], is same as our proposed algorithm except that in evaluating the new values of the dual optimal variables. The numerical evidence confirm that the new proposed algorithm is very effective.

Keyword: sensitivity analysis, GPP, SPP, primal variables, dual variables.

خوارزمية حسابية بديلة لدراسة تحليل الحساسية في مسائل البرمجة المتعددة الحدود السالبة

الملخص

في هذا البحث تم اقتراح خوارزمية حسابية جديدة بديلة لدراسة تحليل الحساسية في مسائل البرمجة الهندسية وبرمجة متعددة الحدود السالبة من ذوات درجة الصعوبة الموجبة. لحل هذا النوع من المسائل نحتاج إلى استخدام روتين فرعي لحل منظومة من المعادلات الخخية لإيجاد القيمة المثلى للمتغيرات في حين أن الخوارزمية المقترحة لا تحتاج إلى حساب معكوس المصفوفة j_{ij}^{-1} وتختلف عن خوارزمية Mohan & Al-Bayati لحساب القيمة الجديدة للمتغيرات المقابلة المثلى. دليل النتائج الحسابية أثبتت كفاءة الخوارزمية المقترحة.

1- Introduction

A geometric programming (GP) is a minimization problem of the form

$$\min g_0(x)=\sum_{k=1}^{n_0}u_k(x)=\sum_{k=1}^{n_0}c_k\prod_{i=1}^{n}x_i^{a_{ij}}$$

$$s.t. \quad g_j(x)\le 1$$

$$and \quad x>0 \quad j=1,...,m$$

where the exponents a_{ij} are arbitrary real numbers, the coefficients c_i are positive and each g_j is a posynomial and are called forced constraints [4]. A signomial programming (SP) is a generalized of a geometric programming, which has the form of a GP, but the objective and constraint functions can be signomials i.e. the coefficients c_k are allowed to be negative [1].On studding the sensitivity analysis we notice that the Theils work [3] for unconstrained problems as well as the DK algorithm [2] and the generalized DK algorithm [1] are essentially based on the assumptions that for a geometric programming (GP) or signomial programming (SP) problems with degree of difficulty greater than zero it is possible to :

(i) Express the solutions to the dual constraints in the form :-

$$di=b_i(0)+\sum_{j=1}^{d}r_j\, b_i(j) \quad i=1,...,n \tag{1}$$

where di are the dual variables, $b_i(j)$ are basis for the column space of the matrix a_{ij} , r_j are certain independent variables and d is the degree of difficulty of the problem.

(ii) The function which give the optimized parameters δ_j^* and $v(\delta^*)$ in terms of the variable coefficient vector *c* are differentiable in an open neighborhood of *c*.

Our work is by rewriting the dual variables and the objective function using some independent dual variables $\delta_{ind.}$ (the variables related with the degree of difficulty).we will find that z(δ) is now essentially a function of the variables δ_i, c_i, and $\delta_{ind.}$ i= m_k, … , n_k, and so we can write

$$z(d) = z(c_j^{'}, d_{ind.})\, where \quad c_j^{'} = s_j\, c_j \, and\, d_{ind.} = d_{ind.1}, d_{ind.2} ..., d_{ind.d} \tag{2}$$

For a given set of values of the variables $c_j^{'}$,there is a set of values $d^*_{ind.1}, d^*_{ind.2} ..., d^*_{ind.d}$ for the variables $d^*_{ind.1}, d^*_{ind.2} ..., d^*_{ind.d}$ for which $z(d)$ is maximum. We can therefore write

$$z(d) = z(c_j^{'}, d^*_{ind.}) \tag{3}$$

where $\delta_{ind.}$ is determined by the set of equations $\dfrac{\partial z}{\partial d^*_{ind.j}} = 0$ j= 1, …, d

At the point of maxima $d^*_{ind.}$ these equations may expressed as

$$f_i(c_j^{'}, d^*_{ind.}) = 0 \qquad i=1,...,d \tag{4}$$

Now suppose the values of the coefficients $c_j^{'}$ are changed to $c_j^{'} + \Delta_j$ then the values of $d^*_{ind.}$ at which the new objective function attains maxima will also change to say $d^*_{ind.} + dd_{ind}$ the new point of maxima will satisfy the conditions:

$$f_i(c_j^{'} + \Delta_j, d^*_{ind.} + dd_{ind.}) = 0 \qquad i = 1, ..., d \tag{5}$$

Now if we suppose that the functions f_i are continuous functions of the variables $c_j^{'}$ & δ_{ind} and posses continuous derivatives in an open neighborhood of $(c_j^{'}, d^*_{ind.})$ then we can expand (5) by Taylor series expansions to $f_i(c_j', d^*_{ind.}) + \left[\Delta_j \dfrac{\partial f_i}{\partial c'_j} + dd_{ind.} \dfrac{\partial f_i}{\partial d_{ind.}}\right] + ... +$ higher order terms $= 0$

where $i = 1,..........,d$. If we now assume that Δ_j and $dd_{ind.}$ are small, then we can neglect the higher order terms of this relation to write :

$$\Delta_j \left[\frac{\partial f_i}{\partial c_j'}\right]_{(c_j^*, d_{ind.}^*)} + dd_{ind.}\left[\frac{\partial f_i}{\partial d_{ind}}\right]_{d_{ind.}^*} = 0 \qquad i = 1,..........,d \qquad (6)$$

Because from (4) $f_i(c_j', d^*_{ind.}) = 0$ .Eqs.(6) is now a system of d linear equations in d unknowns $d\delta_{ind}$. and has a solution of the form:

$$dd_{ind.} = \left[\frac{\partial f_i}{\partial d_{ind.}}\right]^{-1}_{d_{ind.}^*}\left[\frac{\partial f_i}{\partial c_j'}\right]_{(c_j', d_{ind.}^*)} \Delta_j \qquad i = 1,..........,d \qquad (7)$$

Provided $\left[\frac{\partial f_i}{\partial d_{ind.}}\right]^{-1}_{d_{ind.}^*}$ exist, i.e. the matrix $\left[\frac{\partial f_i}{\partial d_{ind.}}\right]_{d_{ind.}^*}$ is non- singular.

Once the increment $dd_{ind.}$ are known we get $d_{ind.}^* + dd_{ind.}$ as the new values of the vector $d_{ind.}$ at which $z(c_j' + \Delta_j, d_{ind.})$ has the maxima. The maximum values of the remain dual variables are now getting by the relations between d_i and $d_{ind.}$ using normality and orthogonality constraint in the dual form the problem .

The new maximum value of the objective function as well as the optimal value of the primal variables can now be obtained as in earlier algorithm in chapter three. It may be observed that in the present analysis our basic assumption is that z is a continuous function of variables $c_j', d_{ind.}$ and can be continuously differentiated in an open neighborhood of $(c_j', d_{ind.})$. This assumption is in no way severe than the assumption (ii) given earlier in this section and which are used in Theils ,DK and generalized DK algorithms.

In the next section we present the explicit form of the computational algorithm for this algorithm which may be used to solve practical problems.

2- A new Computational Algorithm for Solving SPP

Assuming the SPP and its dual in the standard form as follow

$$\begin{aligned}
&\min. \quad g_0(x) = \sum_{i=1}^{n_0} s_i c_i \prod_{j=1}^{m} x_j^{a_{ij}} \\
&s.t. \quad g_k(x) = \sum_{i=m_k}^{n_k} s_i c_i \prod_{j=1}^{m} x_j^{a_{ij}} \le s'_k \\
&c_i > 0, x_j > 0 \qquad (8) \\
&with \\
&s_i \, and \, s'_k = m1
\end{aligned}$$

$$\begin{aligned}
&\max. \quad v(d) = s_0 \left[\prod_{i=1}^{n} \left(\frac{c_i}{d_i} \right)^{d_i} \prod_{k=1}^{p} l_k^{\,l_k} \right]^{s_0} \\
&s.t. \quad \sum_{i=1}^{n_0} s_i d_i = s_0 \\
&\sum_{k=0}^{p} \sum_{i=m_k}^{n_k} s_i a_{ij} d_i = 0 \\
&with \qquad (9) \\
&n_p = n; d_i \ge 0; m_0 = 1; m_1 = n_0 + 1; m_2 = n_1 + 1; ......, m_p = n_{p-1} + 1 \\
&and \\
&l_k = s'_k \sum_{i=m_k}^{n_k} s_i d_i
\end{aligned}$$

We suppose that the optimal solution to the problem is known . In order to carry out the sensitivity analysis for this problem we now proceed as below :-

Step (1) Write

$$z(c'_j, d_{ind.}) = \log\left(v(c'_j, d^*_{ind.})\right) = s_0 \sum_{i=1}^{n} \sum_{k=1}^{p} s_i d_i \log\left(\frac{c_i l_k}{d_i}\right) \qquad (10)$$

And use this to write the system of the equations

$$f_i(c'_j, d^*_{ind.}) = \frac{\partial z}{\partial d_{ind.i}} = 0 \qquad i = 1, ..., d \qquad (11)$$

Step (2) Compute $\Delta d_{ind.}$ for changes Δ_j in $c_j^{'}$ by solving the system of the linear equations

$$\Delta_j \left[\frac{\partial f_i}{\partial c_j^{'}}\right]_{(c_j^{'},d_{ind.}^{*})} + dd_{ind.i}\left[\frac{\partial f_i}{\partial d_{ind.i}}\right] = 0 \qquad i = 1,............,d \tag{12}$$

Step (3) The new maximum value of the objective function is

$$v'(d^{*'}) = v(c_j^{'} + \Delta_j, dd_{ind.}^{*} + dd_{ind.})$$

Step (4) The new optimal values of the primal is

$$g_o(x') = v'(\delta^{*'}) \tag{13}$$

Step (5) The new optimal values of the primal variables are given by solving the system of the following linear equations:

$$(\log x_j^{*'}) = \left[(a_{ij})^T (a_{ij})\right]^{-1} (a_{ij})^T (K_i)^{'}$$

where (14)

$$K_i^{'} = \begin{cases} \log d_i^{*'} + \log v(d^{*'}) - \log(c_i + dc_i), i = 1,........, n_0, \\ \log d_i^{*'} + \log l_k^{*'} - \log(c_i + dc_i), i = n_0 + 1,......., n, \end{cases}$$

and n_0 is the number of terms in the primal objective function.

3- The Effect of the Sensitivity Analysis in GPP

We first apply the new algorithm to solve the following GPP problem (3.1)

$$\min \quad g_0(x) = 2.419 x_1 x_2 x_3 + 95997 x_1^{-1.8} x_3 x_4^{-4.8}$$

$$s.t.$$

$$288670 x_1^{-0.875} x_2^{-0.75} x_3^{-1} x_5^{-0.75} \le 1$$

$$25819 x_1^{-0.2} x_3^{-1} x_4^{0.8} x_6^{-1} \le 1$$

$$0.03866 x_5 + 0.03866 x_6 \le 1$$

$$0.0081666 x_2^{-1} + x_2^{-1} x_4 \le 1$$

$$0.0834 x_2^{-1} \le 1$$

$$x_i > 0; i = 1,2,3,4,5,6$$

This problem has 2 degrees of difficulty . The representation $(d_{ind.})$ for this problem can be written as:

$$d_1 = 0.9554 - 0.24115 d_{ind.1}; d_2 = 0.0446 + 0.2411 d_{ind.1}; d_3 = 1 - d_{ind.1}; d_4 = d_{ind.1};$$
$$d_5 = 0.75 - 0.75 d_{ind.1}; d_6 = d_{ind.1}; d_7 = -0.0089 + 0.1518 d_{ind.1} - d_{ind.2};$$
$$d_8 = 0.2143 + 0.3571 d_{ind.1}; d_9 = d_{ind.2}$$

For the optimal solution

$$d^*_{ind.6} = 0.330558$$
$$d^*_{ind.9} = 0.005169$$

The equations $f_i = \frac{\partial z}{\partial d_{ind.i}} = 0; i = 1,2$ for this problem become :

$$f_1 = \frac{\partial z}{\partial d_{ind.1}} = -.2411 \times \log(c_1) + .2411 \times \log(.9554 - .244 d_{ind.1}) + .2411 \times \log(c_2) -$$
$$02411 \times \log(.0466 + .2411 d_{ind.1}) - \log(c_3) + \log(c_4) - .75 \times \log(c_5) + \log(c_6) +$$
$$.1515 \times \log(c_7) + .3571 \times \log(c_8) + .5089 \times \log(.2054 + .5089 d_{ind.1} - d_{ind.2}) -$$
$$.1518 \times \log(-.0089 + .1518 d_{ind.1} - d_{ind.2}) - .3571 \times \log(.2143 + .3571 d_{ind.1}) +$$
$$.25 \times \log(.75 + .25 d_{ind.1}) - \log(d_{ind.1}) + 0.75 \times \log(.75 - .75 d_{ind.1}) = 0$$

$$f_2 = \frac{\partial z}{\partial \delta_{ind.2}} = \log(c_9) - \log(c_7) + \log(-.0089 + .1618 \delta_{ind.1} - \delta_{ind.2}) -$$
$$\log(.2054 + .5089 \delta_{ind.1} - \delta_{ind.2}) = 0$$

Suppose we want to study the sensitivity analysis for 10% changes in $c_1 = 2.419$ so that $\Delta_1 = 0.2419$ system of equations (6) becomes :

$$\Delta_1 \left[\frac{\partial f_i}{\partial c'_1} \right]_{(c'_1, d^*_{ind.})} + dd_{ind.1} \left[\frac{\partial f_i}{\partial d_{ind.1}} \right]_{d^*_{ind.1}} + dd_{ind.2} \left[\frac{\partial f_i}{\partial d_{ind.1}} \right]_{d^*_{ind.2}} = 0$$

This gives

$$-0.0241102 - 4.9239517 dd^*_{ind.1} + 2.826621951 dd^*_{ind.2} = 0$$

and

$$2.826621951 dd^*_{ind.1} - 25.0052455 dd^*_{ind.2} = 0$$

Solving the above linear system , we get

$dd_{ind.1} = -0.0052363; dd_{ind.2} = 0.000591917$

So at the new optimal solution

$d^{*'}_{ind.1} = 0.3253217; d^{*'}_{ind.2} = 0.004577$

Using step (3) of this algorithm, the maximum value of the objective function is

$v^{'}(d^{*'}) = v(c_1^{'} + \Delta_1, d^*_{ind.1} + dd_{ind.1}, d^*_{ind.2} + dd_{ind.2}) = 202174.78$

The new value of the dual variables are

$\delta^{*'}_1 = .87693$, $\delta^{*'}_2 = .123067$, $\delta^{*'}_3 = .674677$

$\delta^{*'}_4 = .3253217$, $\delta^{*'}_5 = .506008$, $\delta^{*'}_6 = .325322$

$\delta^{*'}_7 = .035873$, $\delta^{*'}_8 = .330472$, $\delta^{*'}_9 = .004577$

The new optimal values of the primal variables are now calculated by using the step (5) of the algorithm .

The result are:

$x^{*'}_1 = 17523.123$, $x^{*'}_2 = .0834$, $x^{*'}_3 = 45.5918$

$x^{*'}_4 = .075233$, $x^{*'}_5 = 15.7443$, $x^{*'}_6 = 10.1222$

Similar analysis was carried out for 50% and 100% percent increases in c_i . The results are presented in table (1).

4- The Effect of the Sensitivity Analysis in SPP

We now apply the new algorithm to solve two SPP which were solved by Mohan & AL-Bayati [1].

Consider the following SP problem (4.1)

min $g_0(x) = 2x_1 x_2^{.5} + x_2 x_3^{-1} x_4^{-2} + x_1^{-2} x_2^{-1} x_3^2$

s.t.

$x_1 x_2^{.5} x_3 - x_2^{-1} x_4^2 \le 1$

$-.2x_1x_2x_3 \le 1$

$x_1x_2x_3x_4 \; > \; 0$

$c_1 = 2$, $c_2 = 1$, $c_3 = 1$, $c_4 = 1$, $c_5 = -1$, $c_6 = .2$

$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 1$, $\sigma_5 = \sigma_6 = -1$, $\sigma'_1 = 1$, $\sigma'_2 = -1$

The degree of difficulty of this problem is 1. For the optimal solution of this problem it is found that $\sigma_0 = 1$

and

$\delta^*_1 = .6952112$, $\delta^*_2 = .1047888$, $\delta^*_3 = .2$, $\delta^*_4 = .1239443$

$\delta^*_5 = .1047888$, $\delta^*_6 = .419554$, $\delta^*_7 = .0191555$

$\delta^*_8 = \delta^*_{ind.} = .4191554$

Also $J(\delta^*) = -39.50795779$ $J^{-1}(\delta^*) = -.025311356$ and for 10% increase in $C_6 = .2$, system of equations (6) becomes

$$\Delta_7 \left[\frac{\partial f_1}{\partial c'_7}\right]_{(c'_7,\delta^*_{ind.1})} + d\delta_{ind.1}\left[\frac{\partial f_1}{\partial f_{ind.s}}\right] = 0$$

This gives

$$(-.02)\left(\frac{4}{.2}\right) + dd_{ind.1}(158.03705) = 0$$

So that $d\,\delta_{ind.1} = .0025311$

So at the new optimal solution $d\delta_{ind.1} = .4216865$. For this new value of $d\delta_{ind.1,}$ the new values for this dual and primal problems were competed by using the appropriate steps of our presented algorithm in section 2. The results are given in Table (4.2). Incremental analysis for 50% change in c_6 was also carried out by this new algorithm. The results are given in the same last table.

Problem (4.2)

$$\min\ g_0(x) = -200x_1^{-1}x_2^{-1} - 500x_1^{-1}x_3^{-\frac{2}{3}} - 820x_1^{-1}x_2^{-\frac{2}{3}}x_3^{-3}x_4^{-1}$$

s.t. $150x_1^{-1}x_2^{-1}x_3^{-1}x_4^{-1} \le 1$

$x_4 + .01x_2x_3x_4 \le 1$

$x_1, x_2, x_3, x_4 > 0$

For this problem

$c_1 = 200$,$c_2 = 500$,$c_3 = 820$,$c_4 = 150$, $c_5 = 1$, $c_6 = .01$

$\sigma_1 = \sigma_2 = \sigma_3 = -1$, $\sigma_4 = 1$, $\sigma_1 = 1$, $\sigma_5 = 1$, $\sigma'_1 = 1$, $\sigma'_2 = 1$

The degree of difficulty of this problem is 1. This problem has negative value for this objective function (i.e. $\sigma_0 = -1$) the optimal solution of this problem was obtained in [1].

At the optimal solution $\delta^*_3 = \delta^*_{ind.1} = .195573$.

For 10% increase in $c_3 = 820$, system (6) becomes

$$(-82)\left(\frac{-1}{-820}\right) + 14.512755 d\delta_{ind.1} = 0$$

There for $d\delta_{ind.1} = .006891$

So at the new optimal solution $\delta^{*'}_{ind.1} = -202463$. The new optimal values for this problem can now be calculated as earlier. Computations were also similarly carried out for 50% change in c_3 . The results are presented in Table (4.3).

We used some brief symbols in the tables of this paper and here are the explanation of those symbols:

- ***MB method*** it is Mohan and AL-Bayati algorithm for studying the sensitivity analysis in SP problems

- ***N. SPP*** it is our new proposed algorithm for studying the sensitivity analysis in GP and SP problems
- ***Inc. An***. it is the method for finding the approximate solution by updating the Jacobian matrix J(x) for reducing that rounding error obtained by MB method

Table (4.1) sensitivity analysis at changes c_1 in problem (3.1)

Original Model	C_1 + 10% solution by		C_1 + 10% solution by			C_1 + 10% solution by	
	MB method	***N. SPP***	***MB method***	***N. SPP***	***Inc. An.***	***MB method***	***N. SPP***
δ_1^* = .875669	.876867	.876930	.880687	.881981	.880917	.88412	Not allowable by this algorithm
δ_2^* = .124331	.123133	.123067	.119313	.118018	.119082	.115876	
δ_3^* = .669442	.674409	.671677	.690257	.695623	.695623	.691209	
δ_4^* = .330558	.325590	..3253217	309743	.304376	.308789	.295487	
δ_5^* = .502081	.505807	.506008	.517693	.521717	.518407	.528384	
δ_6^* = .330558	.325590	.325322	.309743	.304376	.308789	.295487	
δ_7^* = .036076	.035883	.035873	.035268	.035061	.035232	.034716	
δ_8^* = .332342	.330568	.330472	.324908	.322993	.324568	.319817	
δ_1^* = .005169	.004608	.0045771	.002817	.002209	.002708	.001206	
x_1^* = 18054.59	17519.34	17523.12	15884.41	15954.91	15897.02	14504.07	
x_2^* = .0834	.0834	.0834	.0834	.0834	.0834	.0834	
x_3^* = 44.6911	45.5918	45.5918	48.6259	48.8261	48.6775	51.6835	
x_4^* = .075233	.075233	.075233	.075233	.075233	.075233	.075233	
x_5^* = 15.5975	15.7443	15.7443	16.1836	16.3359	13.2106	16.5893	
x_6^* = 10.2690	10.1222	10.1222	9.6829	9.5306	9.6559	9.2773	
g_0 = 185896.41	202174.8	202174.8	265407.6	267288.3	265828.9	342106.3	

Table (4.2) sensitivity analysis of the SPP problem 4.1 for changes in c_6

Original Model	C_6 + 10% solution by		C_6 + 50% solution by		
	MB method	***N. SPP***	***MB method***	***N. SPP***	***Inc. An***
$\delta_1^* = .693211$	.694557	.694579	.691398	.692047	.691586
$\delta_2^* = .104789$	.105443	.105421	.108602	.107953	.108414
$\delta_3^* = .2$	.2	.2	.2	.2	.2
$\delta_4^* = .123944$	.127217	.127109	.143009	.139763	.142074
$\delta_5^* = .104789$	.105443	.105422	.108602	.107953	.108414
$\delta_6^* = .419155$	.421774	.4216858	.434407	.431811	.43366
$x_1^* = 3.05035$	2.90809	2.91312	2.45953	2.54088	2.48201
$x_2^* = .597132$	.605248	2.58485	.643182	.621344	.63698
$x_3^* = 2.74504$	2.58248	2.58485	2.16714	2.14248	2.11692
$x_4^* = 1.80736$	1.71202	1.713958	1.42487	1.45475	1.4331
$g_0 = 6.78108$	6.514723	6.514724	5.70585	5.70621	5.70587

Table (4.3) sensitivity analysis of the SP problem (4.2) for changes in c_3

Original Model	C_3 + 10% solution by		C_3 + 50% solution by		
	MB method	***N. SPP***	***MB method***	***N. SPP***	***Inc. An***
$\delta_1^* = .7$	.7	.7	.7	.7	.7
$\delta_2^* = .104427$	.097961	.097536	.078481	.069975	.07706
$\delta_3^* = .195573$	.202039	.202464	.221519	.230024	0.22294
$\delta_4^* = .1.0$	1.0	1.0	1.0	1.0	1.0
$\delta_5^* = .704427$	.697961	.697536	.678481	.66975	.67706
$\delta_6^* = .1$	.1	.1	.1	.1	.1
$x_1^* = 12.0687$	11.9716	11.9566	11.6793	11.3834	11.6292
$x_2^* = 4.67255$	4.622	4.6299	4.43619	4.5972	4.4624
$x_3^* = 3.03834$	3.10002	3.10237	3.32258	3.39615	3.33402
$x_4^* = .875615$	.874611	.87298	.871477	.83295	.86516
$g_0 = -5.06606$	-5.162996	-5.16297	-5.51836	-5.51037	-5.513774

Summery:

This work deals with the geometric programming problems and the generalized geometric programming problems (SPP) with full rank exponent matrix (a_{ij}) with degree of difficulty greater than zero and less than type of the constraints. We presents a new algorithm to study the sensitivity analysis based on the same of Dinkel and Kohgenberger outlines with some different concepts. Also we checked our algorithm by solving some test problems.

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