Waves and Ripples in Liquid Films
Joseph G. Abdullahad & Abdulrahman M. Morshed

Department of Mathematics
College of Education
University of Mosul

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Abstract

In this paper we present a mathematical model for two-dimensional incompressible flow in a symmetric thin liquid films with the viscosity forces, which can be very small, compared with surface tension and inertia forces. We obtain the governing differential equation for such flow, we also determine the solution of equations and also we consider an inviscid waves in thin films.
1- Introduction:

A problem in fluid mechanics, which received some attention recently, concerns many different types of doubts about the boundary conditions at the surface of liquid films. In the motion of a very thin soap films see for example [5], it has been suggested that an appropriate boundary condition is the kinematic condition of rigidity (inextensible and flexible) rather than the dynamic condition of zero shear stress at the bounding surfaces [1]. Indeed the only existing theory for such motion is a variant of lubrication theory [6], based on this kinematics condition. If, on the other hand, the ordinary zero shear stress condition at the surface is adopted, then the velocity of along the film is almost the same, and this idea is used by [2] when there is a balance between surface tension, inertia and vision forces, and it is also used by [3] for flow in thin liquid films with negligible inertia.

The mechanics of a free surface film flowing steadily between two vertical guide wires was investigated by [4] and shows that the analytic solution can be applied in a loating process.

The objective of this paper is to develop the corresponding theory for more conventional dynamic condition of fluid motion within a symmetric film as shown in figure (1.1) whose flow is in the x. direction of the coordinate axes x and y, and to achieve the derivation of the governing equation for thickness of the liquid film when the flow is inviscid, for both steady and unsteady motion. Furthermore, we consider inviscid waves of small amplitude and the periodic solution of the governing differential equation.
**Waves and Ripples...**

2- **Governing Differential Equations:**

The slope of the film surface is zero on the film proper, and in the transition region remains small approximately in the ratio of its width to a typical value of the radius of curvature, only on the border does the slope reach a substantial values, and thus we take $\partial h/\partial c << 1$ over the domain $x$.

Let $z = \pm h(x, t)$ be the equation of the film surfaces, we assume here a two-dimensional incompressible flow.

Following [2], the governing differential equations of motion of flow within a double sided films are given by:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0,$$

(2.1)

$$\rho\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x}\right) = \sigma \frac{\partial^3 h}{\partial x^3} + 3\mu \frac{\partial^2 u}{\partial x^2},$$

(2.2)

Where $\rho$ is the density of the fluid, $\rho$, the dynamic viscosity, $\sigma$ the surface tension, and $u$ is the component of velocity along the $x$-axis.

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Figure (1.1) Cross section of symmetric film.
3- Inviscid Waves of Small Amplitude:

In the governing differential equations (2.1) and (2.2), if we set

\[ h = h_0 \left( 1 + \varepsilon f(x,t) \right), \] ..........................................................(3.1)

\[ u = \varepsilon g(x,t), \] ..........................................................(3.2)

where \( f, g, f', g', f'', g'' \) are continuous functions and \( \varepsilon << 1 \), then equations (2.1) and (2.2) give after some manipulation, the following fourth order partial differential equation:

\[ \rho \frac{\partial^2 f}{\partial t^2} - 3\mu \frac{\partial^3 f}{\partial x^3 \partial t} + \sigma h_0 \frac{\partial^4 f}{\partial x^4} = 0 \] ...............................................(3.3)

Equation (3.3) has a fundamental solution of the form

\[ f(x, t) = e^{k(ix-\alpha kt)} \] ................................................................(3.4)

Differentiate (3.4) with respect to \( x \) and \( t \), and then substitute these derivatives in equation (3.3), we get

\[ \rho\alpha^2 - 3\alpha\mu + \sigma h_0 = 0 \] ..........................................................(3.5)

If the viscosity (\( \mu \), tends to zero for inviscid approximation, then (3.5) gives

\[ \rho\alpha^2 + \sigma h_0 = 0 \] ..........................................................(3.6)

or

\[ \alpha = \pm i \left( \frac{h_0}{\sigma} \right)^{1/2} \frac{1}{\rho} \]

The fundamental solution (3.4) can be expressed as a linear combination of solutions, yields

\[ f(x, t) = \sin (kx - \alpha kt) \] or

\[ f(x, t) = \sin \left[ k(x - ct) \right] \] .......................................................(3.7)

where \( \alpha = \frac{c}{k}, c \) is the phase velocity, and \( k \) is the wave number.

Now differentiate (3.7) with respect to \( x \) and \( t \), then substitute in equation (3.3), we get
Waves and Ripples...

\[ c = k \left( \frac{\sigma h_0}{\rho} \right)^{1/2} \] ..............................(3.8)

A result which is consistent with the usual theory of propagation of surface tension ripples on a liquid of depth \( h_0 \), and equation (3.8) is comparable with the theory of irrational surface tension waves in a channel of depth \( h_0 \), namely

\[ C = \frac{\sigma k}{\rho} \tanh (kh_0) \] ..............................(3.9)

Provided that \( kh_0 \ll 1 \). This is the usual approximation for long-waves in shallow water, not normally relevant to surface tension waves in channel theory, since such waves are dominated by gravity.

In terms of wave - length \( \lambda \), where

\[ \lambda = \frac{2\pi}{k}, \] ..............................(3.10)

equations (3.8) and (3.10) give the wave speed \( c \),

\[ c = \frac{2\pi}{\lambda} (\tau h_0)^{1/2}, \] ..............................(3.11)

and period

\[ T = \frac{\lambda}{c} = \frac{\lambda^2}{2\pi (\tau h_0)^{1/2}}, \] ..............................(3.12)

Where \( \tau = \frac{\sigma}{\rho} \) is the kinematic surface tension, and it is the only material constant in these inviscid flows.

The following data for \( p \) and \( a \) refer to a standard 20\(^\circ\)C temperature and atmospheric pressure, for \( \lambda=1 \) and for the reliable application of continuum mechanics, a liquid film must be at least 100 molecules thick, we take \( h_0=0.01 \) cm, and then from equations (3.11) and (3.12) we can determine the wave speed \( c \) and the period \( T \) as shown in the following table.
Liquid | Water | Mercury | Glycerin | Carbon tetrachloride | Linseed oil | Olive oil | Turpentine
--- | --- | --- | --- | --- | --- | --- | ---
\(\mu\) gm/cm·sec | 0.0113 | 0.0155 | 14.9 | 0.00974 | 0.4309 | 0.8379 | 0.0149
\(\rho\) gm/cm\(^3\) | 0.998 | 13.55 | 1.26 | 1.59 | 0.94 | 0.91 | 0.86
\(\sigma\) gm/sec\(^2\) | 72.97 | 510.76 | 62.75 | 26.27 | 33.57 | 33.56 | 26.27
\(\tau\) cm\(^3\)/sec\(^2\) | 73.12 | 37.69 | 49.80 | 16.52 | 35.70 | 36.88 | 30.55
\(c\) cm/sec | 5.37 | 3.86 | 4.43 | 2.55 | 3.75 | 3.82 | 3.47
\(T\) sec | 0.186 | 0.259 | 0.226 | 0.392 | 0.266 | 0.266 | 0.288

Table (3.1): The wave speed and the period for various liquids.

4- Sinusoidal Wave:

The theory of inviscid irrational surface waves in a channel of depth \(h_o\) gives:

\[
c^2 = \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right) \tanh(k h_o)
\] ........................................................(4.1)

Now equations (3.8) and (3.12) with equation (4.1) are comparable if:

a) For short wave limit, the wave length \(X\) is very small and \(k h_o\) is very large, but \(\tanh(k h_o)\) is asymptotic to one as \(k h_o\) becomes very large. Thus equation (4.1) gives

\[
c = \left[\frac{g}{k} + \frac{\sigma k}{\rho}\right]^{\frac{1}{2}}
\] ..........................................................(4.2)
Now if \( \lambda_m = L = 2\pi \left( \frac{\sigma}{\rho g} \right)^{1/2} \), ............................................................... (4.3)
then
\[ \lambda << \lambda_m \]
where \( \lambda_m \) denotes the maximum wave length, and so
\[ \lambda << 2\pi \left( \frac{\sigma}{\rho g} \right)^{1/2} \] ................................................................. (4.4)

By using equations (3.10) and (4.4), we get
\[ \frac{\sigma k^2}{\rho g} > 1, \] ................................................................. (4.5)

b) For long wave limit, \( X \) is very large and thus \( kh_o \) is very small, that is
\[ k^2 h_o^2 << 1 \] ................................................................. (4.6)

But \( \tanh(kh_o) \) is asymptotic to \( kh_o \), thus equation (4.1), gives
\[ c^2 = (g + \frac{\sigma k^2}{\rho}) h_o, \] ................................................................. (4.7)

if we neglect the effect of gravity, then equation (4.7) gives
\[ c = k \left( \frac{\sigma h_o}{\rho} \right)^{1/2} \] ................................................................. (4.8)

Which is the same as equation (3.12), such waves in which the only significant restoring force is surface tension, the force responsible for capillary attraction" are often called capillary waves. In water the capillary waves are waves with \( \lambda < 0.4 \) cm, so that it is easy to excite them by striking a tuning fork and placing the tines in the water, see [5]. For example, the tuning fork generates by equation (4.8) wave length \( \lambda \approx 0.171 \) cm, with velocity \( c = 31.42 \) cm/sec.

The short and long wave conditions (4.5) and (4.6) respectively are not incomparable for some liquids.
The value of $L$ from equation (4.3) for water is (1.71 cm) and for mercury is (1.23 cm), other values of $L$ can be determined from equation (4.3), more further, the conditions in term of the wave length $\lambda$, respectively becomes:

a) \[ \frac{\lambda^2}{L^2} << 1 \] .........................................................(4.9)

b) \[ \frac{\lambda^2}{L^2} << \left( \frac{4\pi h_o}{L} \right) \] .........................................................(4.10)

c) \[ \frac{h_o}{H} >> 1, \] .........................................................(4.11)

Where $H = \frac{9\mu^2}{4\sigma\rho}$ .........................................................(4.12)

From table (3.1), it is obvious that the inviscid approximation is relevant water and mercury since the viscosity $\mu$ is very small compared with the other liquids in which this approximation is not relevant to them. The following figure represents the relation between the maximum wave length $\lambda_m$ and the wave speed $c$ for water and mercury.
Waves and Ripples...

\[ c \text{ (cm/sec)} \]

![Graph showing wave length and speed for water and mercury.]

**Figure (4.1):** The wave length \( \lambda \) and the wave speed \( c \) for water and mercury.

From equations (4.3), (4.12), we can determine \( L \) and \( H \) for water and mercury, which are

- \( H_{\text{water}} = 3.94 \times 10^6 \), \( L_{\text{water}} = 1.71 \text{ cm} \)
- \( H_{\text{mercury}} = 7.8 \times 10^8 \), \( L_{\text{mercury}} = 1.23 \text{ cm} \)

The following table gives some values for \( \lambda \) and \( h_o \).

<table>
<thead>
<tr>
<th>Water</th>
<th>Mercury</th>
<th>Water</th>
<th>Mercury</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>( h_o )</td>
<td>( \lambda )</td>
<td>( h_o )</td>
</tr>
<tr>
<td>( \log \left( \frac{\lambda}{L} \right) )</td>
<td>( \log \left( \frac{h_o}{H} \right) )</td>
<td>( \log \left( \frac{\lambda}{L} \right) )</td>
<td>( \log \left( \frac{h_o}{H} \right) )</td>
</tr>
</tbody>
</table>
Table (4.1) values for $\lambda$ and $h_0$ for various liquids.

Now the limit of one percentage error in each approximation gives the following domains of validity of all three approximations in $(h_0, \lambda)$ plane as shown in diagram (4.1).

![Diagram](image)

The separate limits of 1% error in $h_0$, $\lambda$ are:

<table>
<thead>
<tr>
<th>$H_0$ (cm)</th>
<th>$\lambda$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum</td>
<td>Maximum</td>
</tr>
<tr>
<td>$Hg$</td>
<td>$9.01 \times 10^{-6}$</td>
</tr>
</tbody>
</table>
Table

| H₂O  | 3.91 × 10⁻⁴ | 2.77 × 10⁻³ | 2.43 × 10⁻² | 1.72 × 10¹ |

Diagram (4.1): Domains of validity for approximation in (h₀, λ) plane.

5- Conclusion:

The approximation of inviscid waves of small amplitude shows that it is a usual approximation for long waves in shallow water and normally relevant to surface-waves in channel theory since such waves are dominated by gravity. The wave speed c and the period T are evaluated for some liquids, namely water and mercury.

The domain of validity of all approximations in the case of sinusoidal waves when h₀ >> H are investigated, and in this case the only a symptomatic limit h₀ >> H → ∞ is relevant and this limit may be regarded as the limit H → 0, which means that we consider the effect of a very small but non-zero value of viscosity μ.

REFERENCES


