The Error Estimation of Approximating The Crack to Identify The Interfaces Crack in The Mobile's Wire

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ABSTRACT

This paper is devoted to estimate the error that product from approximating crack, i.e. approximating the support of the jump of crack in mobile's wire. This estimation is important to identify the interfaces crack which have been completed by using the derivation of Reciprocity Gap of this model. This derivation is based on the principle of Galerkin finite element method. .

الملخص

هذا البحث مكرس لتخمين الخطأ الناتج من تقريب الشق أو تقريب التحمل للقفز للشق لمطابقة أوجه الشقوق الناتجة في سلك شحن الموبايل . هذه المطابقة تمت من خلال اشتقاق (مبدأ الأساس للثغرة) لهذا النموذج والمبنية على أساس طريقة Galerkin للعناصر المنتهية العددية.

1-Introduction:

This work has focused on the reconstruction of line segment cracks (in 2D situations) or planar cracks (in 3D situations) of a mobile's wire . In this area, there are many theoretical works, and almost all of them deal with 2D cases [8]: a uniqueness result for a buried crack has been investigated by Friedman and Vogelius [6]. For the case of emerging cracks at an a priori known point of the boundary , a uniqueness result and a local Lipschitz stability one have been proved in 1996 by Abda ,Andrieux and Jaoua [1] .In the case of a family of emerging cracks, a uniqueness result has been proved in 2001 by Elcrrat,Isakov and Necoloin [3] .

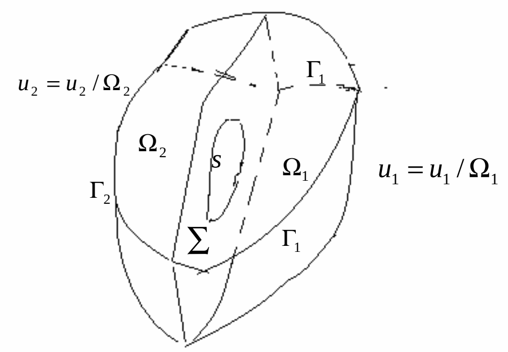
As for the 3D situations , a few uniqueness results exists, and they all assume the knowledge of all the possible measurements , namely the full Neumann-to-Dirichlet operator (see [4] and [5]) .

In this work we will identify the 2D or 3D crack interfaces of mobile's wire by deriving the concept of reciprocity gap of this model which is based on finite element method ,and then we will estimate the error that product from this identification. .

2-The Mathematical Model of mobile's wire and Uniqueness results for 3D planar cracks:

Let Ω denote the 3D bounded domain occupied by the body, and $\partial\Omega$ its external boundary which we shall assume to be C^2. Let Σ be the interface between the two materials in this domain which divided it , Ω into two parts, Ω_i , $i = 1,2$ .The body is supposed to contain one co-planar crack $s \in \Pi$ where Π is the affine plane in R^3 containing the crack .Crack propagated inside the interface as showen in figure (1) .

Figure (1):the representation for part of wire containing crack



The affine space is equipped with a direct orthonormal frame $(0,e_1,e_2,e_3)$. Denoting by (x_1,x_2,x_3) the corresponding Cartesian coordinates system ,the plane equation of the interface will be given by: :

$n_1x_1+n_2x_2+n_3x_3+C=0$ or $N . x + C =0$ -------------(1)

where $N=(n_1,n_2,n_3)$ is unit normal vector to Π on the boundary and on the interface Σ .

Let us denote by f a given heat flux on $\partial\Omega$ satisfying $f \neq 0$ and $\int_{\partial\Omega} f = 0$, $f \in H^{-1/2}(\partial\Omega)$,(in practice , f will be chosen to be piecewise continuous). To solve the problem of conduction (Dirichlet – Neumann boundary value problem (BVP)) :

$$-\nabla(k(x)\nabla u_s) = 0 \qquad \text{in } \Omega \setminus s$$
$$k(x)\frac{\partial u_s}{\partial n} = 0 \qquad \text{on } s \qquad \text{.............(2a)}$$
$$u_s = f \qquad \text{on } \partial\Omega$$
$$k(x)\frac{\partial u_s}{\partial n} = f \qquad \text{on } \partial\Omega$$

We suppose $\int_{\partial\Omega} f = 0$ to ensure the existeness of the solution and we

Assume that $\int_{\partial\Omega} f = 0$ to ensure of the uniqueness and let f* be the trace of

$u_s \; on \; \partial\Omega$ The conductivity k(x) is picewise constant with the int. inside Ω

$$k(x) = \begin{cases} k_1 & \text{in } \Omega_1 \\ k_2 & \text{in } \Omega_2 \end{cases}$$

The discontinuity in k(x) necessitates the consideration of the weak solution

$u_s \in H^1(\Omega)$ of (2a) which satisfies:

$$\int_{\Omega \setminus s} k(x)\nabla u_S \nabla v dxdy = \int_{\partial\Omega} fv.ds + \int_{s} k(s)\frac{\partial u_S}{\partial n} vds + \int_{\partial\Omega} u_S .v = \int_{\partial\Omega} (f + f)vds$$

for all $v \in H^1(\Omega)$ In our case , Σ is the interface between two material and it is sufficiently smooth .

In the classical formulation of (2a) , the solution u_s satisfies the Laplace equation and so-called transmission conditions across the interface : .

$$\left.\begin{aligned} & u_{s1} = u_{s2} \\ & k_1 \frac{\partial u_{s1}}{\partial n} = k_2 \frac{\partial u_{s2}}{\partial n} \end{aligned}\right\} \qquad \text{on} \quad \Sigma \setminus s$$

i.e. continuity of the potential and of the flux. So that :(2a) is equivalent to the two bellow problems (2b), i =1,2.

i.e.The Mathematical model of the Interface of the planar crack in mobile's wire is :

$$\left.\begin{array}{l}
\left.\begin{array}{lll}
k_i \Delta u_{si} = 0 & \text{in} & \Omega_1 / s \\
k_i \dfrac{\partial u_{si}}{\partial n} = 0 & \text{on} & s \\
u_{si} = f & \text{on} & \partial\Omega_1 \cap \partial\Omega = \Gamma_i \\
k_i \dfrac{\partial u_{si}}{\partial n} = f & \text{on} & \partial\Omega_1 \cap \partial\Omega = \Gamma_i
\end{array}\right\} -- \quad for\ i = 1,2 \\
\text{with} \\
\left.\begin{array}{l}
u_{s1} = u_{s2} \\
and \\
k_1 \dfrac{\partial u_{s1}}{\partial n} = k_2 \dfrac{\partial u_{s2}}{\partial n}
\end{array}\right\} \text{----------} \quad \text{on } \Sigma \backslash s
\end{array}\right\} \text{-------------------------} (2b)$$

3-The Derivation of the Reciprocity Gap (the inversion process) for the planar cracks in wire of mobile: .

In this section, we first will derive the numerical principle of the reciprocity gap (inversion process) notion and the functional associated to it using the principles of Galerkin FEM. Then, we use this functional to establish the formulae for the identification and locating the crack interfaces

3-1-The Numerical Principle of Reciprocity gap concept : [7]

In fact this principle is general and is valid in the case of symmetric operators For the sake of simplicity the principle is presented in the case of elliptic operators. The variation formulation associated with this kind of problem can be phrased as follows: .

$$\exists \ \ \text{u in H , such that : } a(u, v) = L(v) \quad \text{for any } v \in H$$

Where H is a Hilbert space, a bilinear , symmetric and coercive form , continuous form on H .L_1 and L_2 being two different linear forms defined on H, let us consider the two corresponding problems (i=1,2):

$$\exists \ \ u_i \text{ in H , such that: } \ a(u_i, v) = L_i(v) \quad \text{for any } v \in H \qquad .$$

Then choosing $v=u_2$ as a test function for the first problem, and $v=u_1$ for the second one, we derive that $L_1(v_2)=L_2(v_1)$. This is the explicit reciprocity principle , due to the symmetry of a. .

3-2-The Numerical Derivation of the Reciprocity Gap concept for esq. (2a) and (2b) :

Lemma(1):

Suppose S is empty and let u be the solution of the problem in the safe wire of mobile :

$$\left.\begin{array}{ll} -\nabla(k(x)\nabla u)=0 & \text{in } \Omega \\ u=f & \text{on } \partial\Omega \\ k(x)\dfrac{\partial u}{\partial n}=f & \text{on } \partial\Omega \end{array}\right\} \text{------------------------}(2c)$$

Then $\forall v$ which is the solution of the equation of conduction: .

$$\nabla(k(x)\nabla v)=0 \qquad \text{in} \quad \Omega \text{ ----------------}(3)$$

we have :

$$RG(v)=\int_{\partial\Omega}[(f+f)v-u(k(s)\frac{\partial v}{\partial n}+v)]=0 \text{ -----------}(4)$$

proof:

that satisfying (3) in Ω and by using the principles of FEM,we have

$$\int_{\Omega}\nabla(k\nabla u)vdx=-\int_{\Omega}k\nabla u\nabla vdx+\int_{\partial\Omega}k\frac{\partial u}{\partial n}vds+\int_{\partial\Omega}fvds$$

$$\int_{\Omega_1}\nabla(k(x)\nabla u)vdx=-\int_{\Omega_1}k_1\nabla u\nabla vdx+\int_{\Gamma_1}k_1\frac{\partial u}{\partial n}vds+\int_{\Gamma_1}fvds+\int_{\Sigma}k_1\frac{\partial u}{\partial n}vds$$

$$=-\int_{\Omega_1}k_1\nabla u\nabla vdx+\int_{\Gamma_1}(f+f)vds+\int_{\Sigma}k_1\frac{\partial u}{\partial n}vds$$

$$\int_{\Omega_2}\nabla(k(x)\nabla u)vdx=-\int_{\Omega_2}k_2\nabla u\nabla vdx+\int_{\Gamma_2}k_2\frac{\partial u}{\partial n}vds+\int_{\Gamma_2}fvds+\int_{\Sigma}k_2\frac{\partial u}{\partial n}vds$$

$$=-\int_{\Omega_2}k_2\nabla u\nabla vdx+\int_{\Gamma_2}(f+f)vds+\int_{\Sigma}k_2\frac{\partial u}{\partial n}vds$$

By the union of the region we have

$$\int_{\Omega_1\cup\Omega_2}\nabla(k(x)\nabla u)vdx=-\int_{\Omega_1\cup\Omega_2}k(x)\nabla u\nabla vdx+\int_{\Gamma_1\cup\Gamma_2}(f+f)vds+\int_{\Sigma\cup\Sigma}k(s)\frac{\partial u}{\partial n}vds$$

also, from (3) we have:

$$\int_{\Omega}\nabla(k(x)\nabla u)vdx=-\int_{\Omega}k(x)\nabla u\nabla vdx+\int_{\partial\Omega}(f+f)vds+\int_{\Sigma}v[k\frac{\partial u}{\partial n}]ds$$

Since $v\in H^1(\Omega)$ satisfying (3).

$$\Rightarrow\int_{\Omega}k(x)\nabla u\nabla v dx=\int_{\partial\Omega}(f+f)v ds+\int_{\Sigma}v[k(s)\frac{\partial u}{\partial n}]ds=\int_{\partial\Omega}(f+f)v ds \text{----------------}(5)$$

Now let $v\in H^1(\Omega)$ be un flux that verify (3),so we have :

$$\Rightarrow\int_{\Omega}k(x)\nabla u\nabla v dx=\int_{\partial\Omega}(k\frac{\partial v}{\partial n}+v).u ds+\underbrace{\int_{\Sigma}u[k(s)\frac{\partial v}{\partial n}]ds}_{=0}$$

since $v\in H^1(\Omega)$ satisfying (3) ,continuous at Σ .

$$\Rightarrow\int_{\Omega}k(x)\nabla u\nabla v dx=\int_{\partial\Omega}k\frac{\partial v}{\partial n}u ds+\int_{\partial\Omega}v.u ds+ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

subtracting (6) from (5) we get:

$$RG(v)=\int_{\partial\Omega}(fv+fv-uk(s)\frac{\partial v}{\partial n}-u.v)ds=0$$

$$\Rightarrow RG(v)=\int_{\partial\Omega}[(f+f)v-u(k(s)\frac{\partial v}{\partial n}+v)]ds=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Hence the proof is completed #

$$\therefore\ RG(v)=\int_{\partial\Omega}[(f+f)v-u(k(s)\frac{\partial v}{\partial n}+v)]ds$$

appears as a linear form defined on the set:

$$H=\{V\in H^1(\Omega)\ such\ that\ \Delta v=0\ \text{in}\ \Omega\}.$$

The following theorem is the key to use of the numerical explicit reciprocity gap functional method for our model: .

Theorem(1):

For $v\in H$ which is the solution of the equation of conduction: .

$$-\nabla(k(x)\nabla v)=0 \qquad \text{in } \Omega\setminus s \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

We have $RG(v)=(\int_{s}[u_s]+\int_{\Sigma\setminus s}[u_s])(k(s)\frac{\partial v}{\partial n}ds)$ where $[u_s]$ denotes the jump of u_s across s and $\Sigma\setminus s$ and u_s is the solution of the problem(2a) or (2b).

Proof:

By applying Green's formula and the principles of Galerkin FEM we have:
For every function v that satisfying (8) in $\Omega\setminus s$ we have:

$$\int_{\Omega\setminus s}\nabla(k\nabla u_s)v dx=-\int_{\Omega\setminus s}k\nabla u_s\nabla v dx+\int_{s}k\frac{\partial u_s}{\partial n}v ds+\int_{\partial\Omega}k\frac{\partial u_s}{\partial n}v ds+\int_{\partial\Omega}u_s v ds$$

$$\int_{\Omega_1\setminus s}\nabla(k\nabla u_{1s})v\,dx=-\int_{\Omega_1\setminus s}k_1\nabla u_{1s}\nabla v\,dx+\int_s k_1\frac{\partial u_{1s}}{\partial n}v\,ds+\int_{\Gamma_1}fv\,ds+\int_{\Gamma_1}fv\,ds+\int_{\Sigma\setminus s}k_1\frac{\partial u_{1s}}{\partial n}v\,ds$$

$$\int_{\Omega_2\setminus s}\nabla(k\nabla u_{2s})v\,dx=-\int_{\Omega_2\setminus s}k_2\nabla u_{2s}\nabla v\,dx+\int_s k_2\frac{\partial u_{2s}}{\partial n}v\,ds+\int_{\Gamma_2}fv\,ds+\int_{\Gamma_2}fv\,ds+\int_{\Sigma\setminus s}k_2\frac{\partial u_{2s}}{\partial n}v\,ds$$

By union the region we have :

$$\underbrace{\int_{\Omega\setminus s}\nabla(k(x)\nabla u_s)v\,dx}_{=0\ \ \text{from}(8)}=-\int_{\Omega\setminus s}k(x)\nabla u_s\nabla v\,dx+\int_s k(s)\frac{\partial u_s}{\partial n}v\,ds+\int_{\partial\Omega}fv\,ds+\int_{\partial\Omega}fv\,ds+\int_{\Sigma\setminus s}v[k(s)\frac{\partial u_s}{\partial n}]ds$$

$$\int_{\Omega\setminus s}k(x)\nabla u_s\nabla v\,dx=\int_s k(s)\frac{\partial u_s}{\partial n}v\,ds+\int_{\partial\Omega}fv\,ds+\int_{\partial\Omega}fv\,ds...(9)$$

Now let $v\in H^1(\Omega\setminus s)$ be un flux that verify (8) so we have:

$$\int_{\Omega\setminus s}k(x)\nabla u_s\nabla v\,dx=\int_s k(s)\frac{\partial v}{\partial n}u_s\,ds+\int_{\partial\Omega}k(s)\frac{\partial v}{\partial n}u_s\,ds+\int_{\partial\Omega}v.u_s\,ds.................................(10)$$

subtracting (10) from (9) we get :

$$\int_s(k\frac{\partial u_s}{\partial n}v-k\frac{\partial v}{\partial n}u_s)ds+\underbrace{\int_{\partial\Omega}[(f+f)vds-u_s(k(s)\frac{\partial v}{\partial n}+v)]ds}_{RG(V)\ \ \text{from}(7)}+\int_{\Sigma\setminus s}(k\frac{\partial u_s}{\partial n}v-k\frac{\partial v}{\partial n}u_s)ds=0$$

$$RG(v)-\int_s[u_s]k(s)\frac{\partial v}{\partial n}ds+\int_{\Sigma\setminus s}(k\frac{\partial u_s}{\partial n}v-k\frac{\partial v}{\partial n}u_s)ds=0$$

$$\therefore RG(v)=\int_s[u_s]k(s)\frac{\partial v}{\partial n}ds+\int_{\Sigma\setminus s}[u_s]k(s)\frac{\partial v}{\partial n}ds=0$$

$$RG(v)=(\int_s[u_s]+\int_{\Sigma\setminus s}[u_s])(k(s)\frac{\partial v}{\partial n}ds)..(11)$$

Remark(1) : (Determine of a normal vector to the interface)

We Denote by x_j the mapping $x\rightarrow x_j$, and let $2L_j=RG(x_j)$ for j=1,2,3.

If f has been chosen in such away that $\int_s[u_s]\ \&\ \int_{\Sigma\setminus s}[u_s]\neq 0$,then the components of the unit normal to the plane (Π)containing the interface that containe crack s are given by:

$$n_j = \frac{L_j}{k\sqrt{L_1^2 + L_2^2 + L_3^2}} \quad \text{for } j = 1,2,3 \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

furthermore, one has : $\left|\int_s [u_s]\right| = \frac{1}{2}\sqrt{L_1^2 + L_2^2 + L_3^2}$

$$\left|\int_{\Sigma\backslash s} [u_s]\right| = \frac{1}{2}\sqrt{L_1^2 + L_2^2 + L_3^2} \ldots\ldots\ldots\ldots\ldots \quad (13)$$

Proposition(1)(Determination of the constant) :

The constant C determining the plane (Π) is given by: .

$$C = \frac{RG(p)}{k\sqrt{L_1^2 + L_2^2 + L_3^2}} \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

where $p(x_1, x_2, x_3) = \frac{x_3^2 - x_2^2}{2}$ ……………………(15)

Proof:

One has $(\int_s [u_s] + \int_{\Sigma\backslash s}[u_s])X_3 = (\int_s [u_s] + \int_{\Sigma\backslash s}[u_s])C$ and $\int_s [u_s]$ & $\int_{\Sigma\backslash s}[u_s]$ is known by equation (11) i.e.

$\int_s [u_s] = \left|\int_s [u_s]\right|$ & $\int_{\Sigma\backslash s}[u_s] = \left|\int_{\Sigma\backslash s}[u_s]\right|$

choosing then $v \in H$ s.t. $\frac{\partial v}{\partial n} = X_3$ on s and appling equation(11) we get:

$$\frac{\partial v}{\partial n} = \frac{RG(v)}{k(\int_s [u_s] + \int_{\Sigma\backslash s}[u_s])} \Rightarrow X_3 = \frac{RG(v)}{k(\left|\int_s [u_s]\right| + \left|\int_{\Sigma\backslash s}[u_s]\right|)}$$

$$\Rightarrow C = \frac{RG(v)}{k(\left|\int_s [u_s]\right| + \left|\int_{\Sigma\backslash s}[u_s]\right|)} \Rightarrow C = \frac{RG(v)}{k(\frac{1}{2}\sqrt{L_1^2 + L_2^2 + L_3^2} + \frac{1}{2}\sqrt{L_1^2 + L_2^2 + L_3^2})}$$

$$\Rightarrow C = \frac{RG(p)}{K\sqrt{L_1^2 + L_2^2 + L_3^2}}$$

Where the polynomial harmonic function $p(x_1, x_2, x_3) = \frac{x_3^2 - x_2^2}{2}$ satisfies the previous conditions.

This mean that one has explicit inversion formulae that give the Cartesian equation of the interface plane containing the crack .

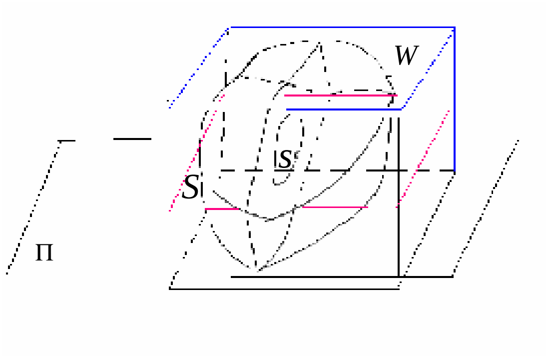
4-The complete identification of the interfaces plane:

In this section , a constructive method is now proposed to achieve the interfaces that contains cracks identification . Once again , the numerical explicit reciprocity gap that we derived in the previous section is a basic tool . Based on its two expressions (7) and (11) , the identification of Σ that containes s is performed by interpreting $[u_s]$ as a linear form of $L^2(S)$, S being some square domain of the plane Π containing the interfaces and the crack . .

Consider now a new frame (O,T,V,N) obtained by a simple translation, such that the new origin O belongs to Π . Let $d = diam(\Omega)$, and w be some open "big box" containing Ω. setting d=2 does not reduce the generality , and then for example: .

$w =]a_1 - 1, a_1 + 1[\times]a_2 - 1, a_2 + 1[\times]a_3 - 1, a_3 + 1[$ Where (a1,a2,a3) are the coordinates of some appropriate interior point to Ω , with respect to the frame (O,T,V,N) as showen in figure (2) . .

Figure (2): the crack laying in the determined plane



Let us choose $S = \Pi \cap W$ and then $S =]-1,1[\times]-1,1[$ after a translation . Define on W the family of functions $(q^i_{p,q})^{i=1,\dots,4}_{p,q\in N}$ as follows : .

$$q^i_{p,q}(x,y,z) = \frac{1}{p\sqrt{p^2+q^2}} x^i_{p,q}(x,y)\sinh(pz\sqrt{p^2+q^2}) \quad \text{.......................................(16)}$$

where $(x^i_{p,q})^{i=1,\dots,4}_{p,q\in N}$ is the orthogonal basis of $L^2(S)$ defined as follows

$$x^1_{p,q}(x,y) = \cos(ppx)\cos(qpy)$$
$$x^2_{p,q}(x,y) = \cos(ppx)\sin(qpy)$$
$$x^3_{p,q}(x,y) = \sin(ppx)\cos(qpy)$$
$$x^4_{p,q}(x,y) = \sin(ppx)\sin(qpy)$$

Lemma(2): [2]

Let $[\tilde{u}_s]$ be the extension by zero of $[u_s]$ to S.then for $p,q \in N$ and i=1,2,3,4

$$RG_{[f,f]}(q^i_{p,q}) = \int_S [\tilde{u}_s] x^i_{p,q} \quad \text{...(17)}$$

Remark(2): [2]

Equation (17) give in fact the Fourier coefficints of $[\tilde{u}_s]$ on the square S and by using a truncated fourier expansion,that is the quadratic partial sum at order n .

$$[\tilde{u}_s]_n = \sum_{p,q=1}^{n}\sum_{i=1}^{4} RG_{[f,f]}(q^i_{p,q})q^i_{p,q} \quad \text{...(18)}$$

we can reconstructing its fourier expansion .

$$RG_{[f,f]}(q^i_{p,q}) = \int_S [u_S]x^i_{p,q} = \int_{\partial\Omega} [f+f]q^i_{p,q} - f(k\frac{\partial q^i_{p,q}}{\partial n} + q^i_{p,q})$$

Now in order to complete the identification of the interfaces crack , it should be estimated the error that product from approximating the support of this jump, i.e. approximating of the crack . .

5-The Error Estimation of approximating the Cracks:

In order to provide an approximation of the cracks, we need to define , for a given positive real number e , and a given integer n , the following sets.

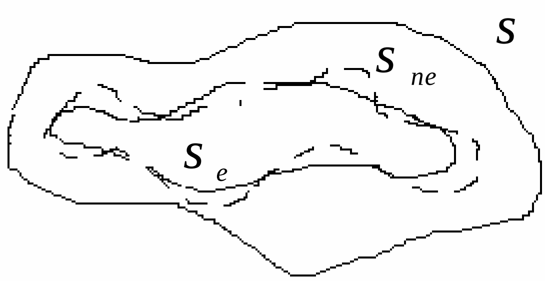
$$s_e = \{x \in S; |[u_s](x)| > e\} \text{......... ....}(19)$$

$$and$$

$$s_{ne} = \{x \in S; |[\tilde{u}_s]_n(x)| > e\} \text{......... ....}(20)$$

The first one is expected to be an approximation of s , and the second one an approximation of s_e. Let us denote by d the Hausdorff distance in the plane (Π) . The following results then holds : .

Figure (3): Approximating of the crack



Lemma(3): [7]

For a prescribed real positive number e , we have:

$$\lim_{n\to\infty} d(s_{ne};s_e)=0 \qquad (21)$$

and furthermore

$$\lim_{e\to 0}\lim_{n\to\infty} d(s_{ne};s)=0 \qquad (22)$$

Lemma(4): [7]

Under the assumption that the distance to the boundary ∂s does not vanish on $\partial\Omega$, there exists some constant c, and some real positive number e_0 depending on $\Omega \setminus s$ and u_s s.t. for any $e \le e_0$ we have $d(s_e;s) \le ce^2$

We are now able to give the error estimate in the following theorem:

Theorem(2):

Under the assumption that the distance to the boundary ∂s does not vanish on $\partial\Omega$,and given any positive number d , there exists some constant $\tilde{c},\tilde{k}$, and some real positive number e_0 s.t. for any $e \le e_0$ and $n > \tilde{c}e^{-(2+d)}$ we have:

$$d(s_{ne};s)=\max\left|(g-g_n)(x)\right| \le \tilde{k}.e$$

Proof:

We shall use the following characterization of the sobolev space $H^s(S)$ [9]

$$H^s(S)=\{g\in L^2(S);\sum_{p,q=1}^{\infty}\sum_{i=1}^{4}(1+p^2+q^2)^s\left|RG_{[f,f]}(q^i{}_{p,q})\right|^2<\infty\}$$

where $RG_{[f,f]}(q^i{}_{p,q})$ are the fourier coefficients of g , the space $L^2(S)$ being the set of functions g verifying $\sum_{p,q=1}^{\infty}\sum_{i=1}^{4}\left|RG_{[f,f]}(q^i{}_{p,q})\right|^2<\infty$.

For $g\in H^s(S)$ and $s=1-g, g<1$,we have :

$$[\tilde{u}_s]_n=(g-g_n)(x)=\sum_{p,q>n}\sum_{i=1}^{4}RG_{[f,f]}(q^i_{p,q}).q^i_{p,q}$$

$$(g-g_n)(x)=\sum_{p,q>n}\sum_{i=1}^{4}(\int_{\partial\Omega}[f+f]q^i_{p,q}-f(k\frac{\partial q^i_{p,q}}{\partial n}+q^i_{p,q})).q^i_{p,q}$$

which can also be written:

$$(g-g_n)(x)=\sum_{p,q>n}\sum_{i=1}^{4}(1+p^2+q^2)^{\frac{-s}{2}}(1+p^2+q^2)^{\frac{s}{2}}\Big(\int_{\partial\Omega}[f+f]q^{i}_{p,q}-f\big(k\frac{\partial q^{i}_{p,q}}{\partial n}+q^{i}_{p,q}\big)\Big).q^{i}_{p,q}$$

Using the Cauchy –Schwarz inequality ,we get:

$$\max\left|(g-g_n)(x)\right|\le\{\sum_{p,q>n}(1+p^2+q^2)^{-s}\}^{1/2}.$$

$$.\{\sum_{p,q>n}\sum_{i=1}^{4}(1+p^2+q^2)^{s}\Big(\int_{\partial\Omega}[f+f]q^{i}_{p,q}-f\big(k\frac{\partial q^{i}_{p,q}}{\partial n}+q^{i}_{p,q}\big)\Big)^2\}^{1/2}$$

since $RG_{[f,f]}(q^{i}_{p,q})$ are the fourier coefficients of g, we get:

$$\max\left|(g-g_n)(x)\right|\le\{\sum_{p,q>n}(1+p^2+q^2)^{-s}\}^{1/2}.\underbrace{\{\sum_{p,q>n}\sum_{i=1}^{4}(1+p^2+q^2)^{s}\left(\left|RG_{[f,f]}(q^{i}_{p,q})\right|\right)^2\}^{1/2}}_{\to 0}$$

Since $g\in H^s(S)$,the second term between brackets is converging to 0 , and is then bounded by some k1>0 . Thus for any s' smaller than s , we derive:

$$\max\left|(g-g_n)(x)\right|\le$$
$$\le k_1\{\sum_{p,q>n}(1+p^2+q^2)^{-s}(1+p^2+q^2)^{-s'}(1+p^2+q^2)^{s'}\}^{1/2}$$

$$\max\left|(g-g_n)(x)\right|\le$$
$$\le k_1\{\sum_{p,q>n}(1+p^2+q^2)^{-s'}(1+p^2+q^2)^{-(s-s')}\}^{1/2}$$

$$\max\left|(g-g_n)(x)\right|\le$$
$$\le k_1\Big(\sum_{p,q>n}(1+p^2+q^2)^{-s'/2}\Big)\Big(\sum_{p,q>n}(1+p^2+q^2)^{-(s-s')}\}^{1/2}$$
$$\le\frac{k_1}{\sum_{p,q>n}(1+p^2+q^2)^{s'/2}}\{\sum_{p,q>n}(1+p^2+q^2)^{-(s-s')}\}^{1/2}$$
$$\le\frac{k_1}{(1+2n^2)^{s'/2}}\{\sum_{p,q>n}(1+p^2+q^2)^{-(s-s')}\}^{1/2}$$

Now since any choice of s' s.t. $s-s'>\frac{1}{2}$ would insure the convergence of the series between brackets, therfore it is bounded by M, i.e.

$$\max\left|(g-g_n)(x)\right| \leq \frac{k_1 M}{(1+2n^2)^{s'/2}} \leq \frac{k_1 M}{(2n[\frac{1}{2n}+n])^{s'/2}} \leq \frac{k_1 M}{(2n)^{s'/2}[\frac{1}{2n}+n]^{s'/2}} \leq$$

$$\leq \frac{k_1 M}{(n)^{s'/2}}(2[\frac{1}{2n}+n])^{-s'/2} \leq \frac{k_1 M}{(n)^{s'}}.n^{-1/2}(2[\frac{1}{2n}+n])^{-s'/2} \leq$$

$$\leq \frac{k_1 M}{(n)^{s'}}.n^{-1/2}[(\frac{1}{n}+2n)^{-1/2}]^{s'} \leq$$

$$\leq \frac{k_1 M}{(n)^{s'}}.[n^{1/s'}(\frac{1}{n}+2n)\ ]^{-s'/2}$$

Now by taking $s' = \frac{1-2g'}{2}$ with $g < g' < \frac{1}{2}$ we obtain:

$$\max\left|(g-g_n)(x)\right| \leq \frac{k_1 M}{(n)^{\frac{1}{2}-g'}}.[n^{\frac{1}{\frac{1-2g'}{2}}}(\frac{1}{n}+2n)\ ]^{-(\frac{1-2g'}{4})}$$

$$\leq \frac{k_1 \tilde{k}}{(n)^{\frac{1}{2}-g'}}$$

where $\tilde{k} = M.[n^{\frac{1}{\frac{1-2g'}{2}}}(\frac{1}{n}+2n)\ ]^{-(\frac{1-2g'}{4})}$

Now in order to get $\left|(g-g_n)(x)\right| \leq \tilde{k}.e$ it is sufficient to choose :

$$\frac{k_1}{(n)^{\frac{1}{2}-g'}} = k_1.(n)^{-\frac{1-2g'}{2}} \leq e \Rightarrow e^{\frac{2}{1-2g'}} \geq (k_1)^{\frac{2}{1-2g'}}.n^{-\frac{1-2g'}{2}(\frac{2}{1-2g'})}$$

$$\Rightarrow e^{-\frac{2}{1-2g'}} \geq (k_1)^{-\frac{2}{1-2g'}}.n^{-1}$$

$$\Rightarrow ne^{-\frac{2}{1-2g'}} \geq (k_1)^{-\frac{2}{1-2g'}}$$

$$\Rightarrow n \geq \frac{(k_1)^{+\frac{2}{1-2g'}}}{e^{\frac{2}{1-2g'}}}$$

this implies that :

$$N(e) \geq \tilde{c}\, e^{-\frac{2}{1-2g'}}, \quad where\ \tilde{c} = (k_1)^{\frac{2}{1-2g'}}$$

so that it is sufficient to choose $N(e) \geq \tilde{c}\, e^{-\frac{2}{1-2g'}}, \quad where\ \tilde{c} = (k_1)^{\frac{2}{1-2g'}}$ to get $\left|(g - g_n)(x)\right| \leq \tilde{k}.e$

Now let $g' = \dfrac{d}{2(2+d)}$, the condition on $N(e)$ becomes then:

$$N(e) \geq \tilde{c}\, e^{-\frac{2}{1-2g'}} \Rightarrow N(e) \geq \tilde{c}\, e^{-\frac{2}{1-2\left(\frac{d}{2(2+d)}\right)}} \Rightarrow N(e) \geq \tilde{c}\, e^{-(2+d)}$$

Now according to Zhizhiashvili, the uniform convergence of g_n to g . for any given e ,there exist $N(e)$ s.t. .

$$n \geq N(e) \Rightarrow \max_{x \in S}\left|g_n(x) - g(x)\right| \leq e$$

So that choosing $n \geq \tilde{c}\, e^{-(2+d)}$ insures that : $\max_{x \in S}\left|(g_n - g)(x)\right| \leq \tilde{k}.e$ and according to the proof of lemma(3) and(4) [7] we conclude that:

$$d(s_{ne}; s) = \max\left|(g - g_n)(x)\right| \leq \tilde{k}.e$$

Hence the proof is complete #

6-Conclusion:

The Rreciprocity Gap concept derived in this paper seems to be quite efficient, both from the theoretical and the numerical viewpoints. It leads to uniqueness results for the planar crack inverse problem as well as explicit inversion formulae for the interface containing the cracks. We have been introduced and derived this concept to identify the interfaces of crack in the mobile's wire and complete this identification by estimating the error of approximating this cracks .

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