Rings in which Every Simple Right
R-Module is Flat

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ABSTRACT

The objective of this paper is to initiate the study of rings in which each simple right R-module is flat, such rings will be called right SF-rings. Some important properties of right SF-rings are obtained. Among other results we prove that: If R is a semi prime ERT right SF-ring with zero socle, then R is a strongly regular ring.
1. Introduction:

Throughout this paper, R denotes associative ring with identity and all modules are unitary. \( J( R ) \) and \( Y( R ) \) denote the Jacobson radical and the singular right ideal of \( R \), respectively. For any nonempty subset \( X \) of a ring \( R \), the right (resp. left) annihilator of \( X \) will be denoted by \( r(X) \) (resp. \( l(X) \)). Recall that:

1. A ring \( R \) is ERT if every essential right ideal of \( R \) is a -sided,
2. \( R \) is said to be von Nuemann regular (or just regular) if,
   \[
   a \in aRa, \text{ for every } a \in R, \text{ and } R \text{ is called strongly regular if } a \in a^2 R.
   \]

In [3] Ming asked the following question:

Is a semi prime right SF-ring, all of whose essential right ideals are two-sided von Neumann regular?

In this paper, we give conditions for a semi prime right SF-ring all of whose essential right ideals are two sided to be von Neumann regular.

2. Basic Properties:

Following [5], a ring \( R \) is called a right (left) SF-ring, if every simple right (left) \( R \)-module is flat.

The following lemma which is due to [7], plays a central role in several of our proofs: Lemma 2.1:
Let I be a right ideal of R. Then R/I is a flat R-module if and only if for each ae I, there exists be I such that a=ba.

We shall begin with the following result:

**Proposition 2.2:**

If R is a right SF-ring. Then

1. Any reduced principal right ideal of R is a direct summand.
2. Every left or right R-module is divisible.

**Proof (1):**

Let I=aR be a reduced principal right ideal of R and let aR+r(a) ≠ R. Then there exists a maximal right ideal M of R containing aR+r(a).

Now, since R/M is flat, then a=ba, for some b in R. Whence l-b \in l(a)=r(a) ≤ M. Yielding l \in M which contradicts M*R. In particular

ar+c=l, for some r \in R and c \in r(a), whence a^2 r=a. If we set d=ar^2 \in l, then a=a^2d. Clearly, (a-ada)^2=0 implies a=ada and hence I=eR, where e=ad, is idempotent element. Thus I is a direct summand.

**Proof (2):**

It is sufficient to prove that any non-zero divisor c of R is invertible. For then, if dc=cd=l, any right R-module M satisfies M=Mdc⊆Mc⊆M, whence M=Mc (similarly, any left
R-module is divisible). Suppose that \( cR \neq R \). Let \( K \) be a maximal right ideal containing \( cR \). Since \( R/K \) is flat, there exist \( u \in K \), such that \( c = uc \). Now, \( r(c) = 1 \) \( (c) = 0 \) implies \( u = 1 \), contradicting \( K \neq R \).

This proves that \( cd = 1 \) for some \( d \in R \) and hence \( dc = 1 \).

**Proposition 2.3:**

Let \( R \) be a right SF-ring. Then either \( r(M) = 0 \) or \( M \) is a direct summand.

**Proof:**

Suppose that \( r(M) \neq 0 \) and let \( b \in M \cap r(M) \). Then \( b \in M \) and \( Mb = 0 \). Since \( R/M \) is flat then there exists \( a \in M \) such that \( b = ab \). Now \( b = ab \in Mb = 0 \), so \( b = 0 \). Thus \( M \cap r(M) = 0 \), this means that \( M \) not can be essential and hence \( M \) is a direct summand. Therefore \( r(M) \oplus M = R \).

**3. The Connection Between SF-Rings and Other Rings:**

In this section we study the connection between SF-rings, biregular rings and strongly regular rings.

Recall that the right (left) socle of a ring \( R \) is defined to be the sum of all minimal right (left) ideals of \( R \). It is well know that in a semi prime ring \( R \), the right and left socles of \( R \) coincide, which will be denoted by \( \text{soc}R \).
Following [4], a ring $R$ is biregular if $RaR$ is generated by a central idempotent for each $a \in R$.

**Theorem 3.1:**

Let $R$ be an ERT SF-ring with right zero socle and for every $a \in R$, $RaR$ is a principal right of $R$. Then $R$ is biregular.

**Proof:**

For any $a \in R$, set $M=RaR+\l(RaR)$. Since $M$ is a maximal right ideal, then $M$ is a direct summand or essential. If $M$ is a direct summand of $R$, then its complement is a minimal right ideal. This implies that $R$ has a no-zero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided. By hypothesis $R/M$ is flat. Also $RaR=bR$ for some $b \in R$. Since $b \in M$, $b=db$ for some $d \in M$. Then $l-d \in l(b) \cap M$ which yields $l \in M$. Therefore $l=bc+v$, $c \in R$, $v \in l(b)$ this implies that $b=bcb$. Therefore $RaR=bR=eR$ where $e=bc$ is idempotent. $R$ is therefore semi-prime and hence $e$ is central in $R$. Thus $R$ is biregular.

**Theorem 3.2:**

If $R$ is a reduced ring and every maximal right ideal of $R$ is either a right annihilator or flat, then $R$ is strongly regular.
Proof:

Let \( b \in R \). We claim first \( bR + r(b) = R \). If not, there exists a maximal right ideal \( L \) containing \( bR + r(b) \). In case \( R/L \) is flat, since \( b \in L \), there exists \( c \in L \) such that \( b = cb \). Then \( l - c \in l(b) = r(b)cL \), whence it follows \( 1 \in L \), a contradiction. On the other hand, in case \( L = r(t) \) with some \( 0 \neq t \in R \), we have \( t \in !(bR + r(b)) \subseteq l(b) = r(b) \subseteq L = r(t) \). Then \( t^2 = 0 \), a contradiction. Therefore let \( bR + r(b) = R \), and hence \( R \) is strongly regular.

Now, we give under what condition the answer of the question of ring is affirmative.

Proposition 3.2:

Let \( R \) be a semi-prime ERT right SF-ring with zero socle. Then \( R \) is strongly regular.

Proof:

Let \( M \) be a maximal right ideal of \( R \). Then \( M \) is either a direct summand of \( R \) or an essential right ideal of \( R \). If \( M \) is a direct summand of \( R \), then its complement is a minimal right ideal. This implies that \( R \) has a nonzero socle, which is a contradiction. So every maximal right ideal is essential and hence two-sided.

Since \( R \) is semi-prime ERT, by applying [2,Lemma 2.1], we see that \( R \) is right non-singular, and \( J(R) = 0 \). So \( R \) is isomorphic to a sub direct sum of division rings, which implies that \( R \) has no
Theorem 3.3:

Let $R$ be a prime ERT and right SF-ring. Then $R$ has non-zero socle.

Proof:

Let $R$ be a prime ERT and right SF-ring. If $\text{Soc} R = 0$. By the above Theorem (3.2), $R$ is strongly regular. Hence $R$ is a division ring, and $\text{Soc} R = R$ contradicting our assumption. Therefore $\text{Soc} R \neq (0)$.

REFERENCES