The n-Wiener Polynomials of Straight Hexagonal Chains and $K_r \times C_r$

Ali A. Ali
College of Computer Sciences and Mathematics
University of Mosul

Haveen G. Ahmed
College of Science
University of Dohuk.

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ABSTRACT

The n-Wiener polynomials of straight hexagonal chains and the Cartesian product of a complete graph $K_r$ and a cycle $C_r$ are obtained in this paper. The n-diameter and the n-Wiener index of each such graphs are also determined.

Keywords: n-distance, n-diameter, n-Wiener index, n-Wiener Polynomials, hexagonal chains.

1. Introduction.

We follow the terminology of [5,6]. Let $v$ be a vertex of a connected graph $G$ and let $S$ be an $(n-1)$-subset of vertices of $V(G)$, $n \geq 2$, then the n-distance $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S) = \min \{d(v,u) : u \in S \}. \quad \text{...(1.1)}$$

Sometimes, we refer to the n-distance of the pair $(v,S)$ in $G$ by $d_n(v,S \mid G)$.

The n-diameter $\text{diam}_n G$ of $G$ is defined by

$$\text{diam}_n G = \max \{d_n(v,S) : v \in V(G), S \subseteq V(G), |S| = n-1 \}. \quad \text{...(1.2)}$$

It is clear that for all $2 \leq m \leq n \leq p$,

$$\text{diam}_n G \leq \text{diam}_m G \leq \text{diam} G. \quad \text{...(1.3)}$$

The n-Wiener index of $G$ denoted by $W_n(G)$ is defined as

$$W_n(G) = \sum_{(v,S)} d_n(v,S), \quad \text{...(1.4)}$$

where the summation is taken over all pairs $(v,S)$ for which $v \in V(G)$, $S \subseteq V(G)$ and $|S| = n-1$. The n-average distance $\mu_n(G)$ is defined as
\( \mu_n(G) = \frac{W_n(G)}{p \binom{p-1}{n-1}}, \quad 3 \leq n \leq p. \) \hspace{1cm} ...(1.5)

Let \( v \) be any vertex of \( G \), then the **n-distance of** \( v \) denoted \( d_n(v \mid G) \) or simply \( d_n(v) \) is defined as
\[
\begin{align*}
d_n(v) &= \sum_{S \subseteq V(G)} d_n(v, S), \quad |S| = n-1. \tag{1.6}
\end{align*}
\]

The Wiener polynomial of \( G \) with respect to the n-distance, which is called n-Wiener polynomial and defined as below.

**Definition 1.1** [2]. Let \( C_n(G,k) \) be the number of pairs \( (v, S) \), \( |S| = n-1, 3 \leq n \leq p \), such that \( d_n(v,S)=k \), for each \( 0 \leq k \leq \delta_n \). Then, the **n-Wiener polynomial** \( W_n(G;x) \) is defined by
\[
W_n(G;x) = \sum_{k=0}^{\delta_n} C_n(G,k)x^k, \hspace{1cm} ...(1.7)
\]
in which \( \delta_n \) is the n-diameter of \( G \).

One may easily see [2] that for \( 3 \leq n \leq p \), the number of all \( (v,S) \) pairs is
\[
p \binom{p-1}{n-1}, \hspace{1cm} \text{and}
\]
\[
\sum_{k=1}^{\delta_n} C_n(G,k) = p \binom{p-1}{n-1}, \quad C_n(G,0) = p \binom{p-1}{n-2}, \hspace{1cm} ...(1.8)
\]
\[
C_n(G,1) = p \binom{p-1}{n-1} - \sum_{v \in V(G)} \binom{p-1}{n-1} \deg_G(v). \hspace{1cm} ...(1.9)
\]

**Definition 1.2** [1] Let \( v \) be a vertex of a \( G \), and let \( C_n(v,G,k) \) be the number of \( (n-1) \)-subsets of vertices of \( G \) such that \( d_n(v,S \mid G) = k \), for \( n \geq 3 \), \( 0 \leq k \leq \delta_n \). Then, the **n-Wiener polynomial of vertex** \( v \), denoted by \( W_n(v,G;x) \) is defined as
\[
W_n(v,G;x) = \sum_{k=0}^{\delta_n} C_n(v,G,k)x^k. \hspace{1cm} ...(1.10)
\]

It is clear that for all \( k \geq 0 \),
\[
\sum_{v \in V(G)} C_n(v,G,k) = C_n(G,k), \hspace{1cm} ...(1.11)
\]
and
\[
\sum_{v \in V(G)} W_n(v,G;x) = W_n(G;x). \hspace{1cm} ...(1.12)
\]
There are many classes of graphs $G$ in which for each $k, 1 \leq k \leq \delta_n$, $C_n(v, G, k)$ is the same for every vertex $v \in V(G)$; such graphs are called [1] **vertex-n-distance regular**. If $G$ is of order $p$ and it is vertex-n-distance regular, then

$$W_n(G; x) = pW_n(v, G; x), \quad \text{...(1.13)}$$

where $v$ is any vertex of $G$.

The authors of papers [2,3,4] obtained the n-Wiener polynomials and n-Wiener index for some special graphs and of some kind of compound graphs. In this paper, we obtain n-Wiener polynomials for straight hexagonal chains and for the Cartesian product $K_t \times C_r$.

**2. The Cartesian Product of a Cycle and a Complete Graph**

Let $C_r$ be a cycle of order $r \geq 3$ and vertices $v_1, v_2, \ldots, v_i, v_1$, and let $K_t$ be a complete graph of vertex set $V(K_t) = \{u_1, u_2, \ldots, u_t\}$. It is clear that $K_t \times C_r$ is regular of degree $t+1$, and it is vertex-n-distance regular. Thus, for every vertex $(u_i, v_j)$ of $K_t \times C_r$ and each $k$

$C_n((u_i, v_j), K_t \times C_r, k)$

has the same value for $2 \leq n \leq tr$. Therefore,

$$diam_n K_t \times C_r = \max\{d_n(u_1, v_1): S \subseteq V(K_t \times C_r), |S| = n-1\}.$$

The $n$-diameter of $K_t \times C_r$ is determined in the next proposition.

**Proposition 2.1.** For $r = 2s$, $s \geq 2$, $t \geq 3$,

$$diam_n K_t \times C_r = s + 1 - \left\lfloor \frac{(n+t-1)}{2t} \right\rfloor,$$

when $2 \leq n \leq tr$.

**Proof.** Let $A_i = \{(u_j, v_i): j = 1, 2, \ldots, t\}, 1 \leq i \leq r$. The induced subgraph $<A_i>$ is denoted by $K_t^{(i)}$ and called the $i$th copy of $K_t$. It is clear that $diam K_t \times C_r = s+1$,

therefore, for $2 \leq n \leq tr$

$$diam_n K_t \times C_r \leq s+1.$$

Now if $2 \leq n \leq t$, then we take $S \subseteq A_{s+1} - \{(u_1, v_{s+1})\}$, and we find that $d_n(u_1, v_1, S) = s+1$,

as it is clear from $K_t \times C_r$ shown in Fig.2.1. Thus,

$$diam_n K_t \times C_r = s+1,$$

when $2 \leq n \leq t$.

Now, assume that $t+1 \leq n \leq tr$. 

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Fig. 2.1. The graph $K_t \times C_{2s}$

If $\mathcal{S}$ is an $(n-1)$-set of vertices such that $d_n((u_1,v_1),\mathcal{S})$ is maximum, then $\mathcal{S}$ must contain $A_{s+1}$, and the other $n-t-1$ vertices are taken from the set

\[\{(A_s \cup A_{s+2}) \cup (A_{s+1} \cup A_{s+3}) \cup \ldots \cup (A_{s+1,j} \cup A_{s+j+1})\} - \{(u_1,v_{s+1-j}), (u_1,v_{s+j+1})\}\]

such that

\[t+2t(j-1) \leq n-1 \leq 3t-2+2t(j-1)\]

Solving for $j$, we get

\[(n-t+1)/2t \leq j \leq (n+t-1)/2t.\]

Since $j$ is an integer, we have

\[j = \left\lfloor \frac{(n+t-1)}{2t} \right\rfloor.\]

From Fig. 2.1, one can easily see that

\[d_n((u_1,v_1),\mathcal{S}) = s+1-j.\]

Hence the proof is completed.  

**Proposition 2.2.** For $r=2s+1$, $s \geq 1$, $t \geq 3$,

\[\text{diam}_h(K_t \times C_{2r}) = s+1 - \left\lfloor \frac{n}{2t} \right\rfloor, \text{ for } 2 \leq n \leq tr.\]

**Proof.** Consider the graph $K_t \times C_{2s+1}$ shown in Fig. 2.2 and use the notations used in the proof of Proposition 2.1.
Let $S$ be an $(n-1)$-set of vertices such that $d_n((u_1,v_1),S)$ is maximum. If $2 \leq n \leq 2t-1$, then $S \subseteq A_{s+1} \cup A_{s+2} - \{(u_1,v_{s+1}),(u_1,v_{s+2})\}$, and 

$$d_n((u_1,v_1),S) = s+1 = \text{diam}_n K_t \times C_r,$$

as given in the proposition. 

If $2t \leq n \leq rt$, then $S$ must consist of vertices from 

\[
[(A_{s+1} \cup A_{s+2}) \cup (A_s \cup A_{s+3}) \cup \ldots \cup (A_{s+1+j} \cup A_{s+j+2})] - \{(u_1,v_{s+1-j}),(u_1,v_{s+j+2})\}
\]

such that 

$$2tj-1 \leq n-1 \leq 2t(j+1)-2 \ (\text{See Fig. 2.2}).$$

Solving for $j$, we get 

$$ (n/2t) - (2t-1)/2t \leq j \leq n/2t. $$

Since $j$ is a positive integer, then $j = \lceil n/2t \rceil$. It is clear from the figure that 

$$d_n((u_1,v_1),S) = s+1-j.$$

Therefore, 

$$\text{diam}_n K_t \times C_{2s+1} = s+1-j = s+1-\lceil n/2t \rceil.$$

We determine the $n$-Wiener polynomial of $K_t \times C_r$ in the following theorems.

**Theorem 2.3.** For $t \geq 2$, $3 \leq n \leq rt$, $r=2s+1$, we have 

$$W_n(K_t \times C_r;x) = \sum_{k=0}^{rt} C_n(K_t \times C_r,k)x^k,$$

in which 

$$C_n(K_t \times C_r,0) = \binom{rt-1}{n-2}.$$
\[ C_n(K_t \times C_r, 0) = rt \left( \binom{n-1}{n-1} - \binom{n-2}{n-1} \right), \]

and for \( 2 \leq k \leq s, \)
\[ C_n(K_t \times C_r, k) = rt \left( \binom{\alpha + 2i}{n-1} - \binom{\alpha}{n-1} \right), \]

where
\[ \alpha = 2t(s-k+1) - 2, \]
\[ C_n(K_t \times C_r, s+1) = rt \left( \binom{2r-2}{n-1} \right), \]

and \( \delta_n \) is the n-diameter of \( K_t \times C_r. \)

**Proof.** \( C_n(K_t \times C_r, 0) \) and \( C_n(K_t \times C_r, 1) \) follow from (1.8) and (1.9). Since \( K_t \times C_r \) is vertex-n-distance regular, we have for \( 2 \leq k \leq \delta_n, \)
\[ C_n(K_t \times C_r, k) = rtC_n((u_1, v_1), K_t \times C_r, k) \] (See Fig. 2.2).

For \( 2 \leq k \leq s \) there are \( 2t \) vertices each of distance \( k \) from vertex \( (u_1, v_1), \) and there are \( 2t(s-k+1) - 2(=\alpha) \) vertices each of distance more than \( k \) from \( (u_1, v_1). \) Thus,
\[ C_n((u_1, v_1), K_t \times C_r, k) = \sum_{j=0}^{n-1} \binom{2t}{n-1-j} \binom{\alpha}{j}, \quad 2 \leq k \leq s. \]

If \( 3 \leq n \leq 2t-1, \) then \( \delta_n = s+1. \) In this case, there are exactly \( 2t-2 \) vertices each of distance \( s+1 \) from vertex \( (u_1, v_1), \) and there is no vertex of distance more than \( s+1. \) Therefore,
\[ C_n((u_1, v_1), K_t \times C_r, s+1) = \binom{2t-2}{n-1}. \]

**Theorem 2.4.** For \( t \geq 2, \) \( 3 \leq n \leq rt, \) \( r = 2s \geq 4, \) we have
\[ W_n(K_t \times C_r; x) = \sum_{k=0}^{\delta_n} C_n(K_t \times C_r, k)x^k, \]

where \( \delta_n \) is the n-diameter, and
\[ C_n(K_t \times C_r, 0) = rt \left( \binom{n-1}{n-2} \right), \]
\[ C_n(K_t \times C_r, 1) = rt \left( \binom{n-1}{n-2} - \binom{n-2}{n-1} \right), \]

and for \( 2 \leq k \leq s-1, \)
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\[ C_n(K_t \times C_r, k) = rt\left( \frac{\beta + 2r}{n-1} \right) \left( \frac{\beta}{n-1} \right), \beta = t(r - 2k + 1) - 2, \]

\[ C_n(K_t \times C_r, s) = rt\left( \frac{3r - 2}{n-1} \right) \left( \frac{\tau - 1}{n-1} \right), \]

\[ C_n(K_t \times C_r, s+1) = rt\left( \frac{\tau - 1}{n-1} \right), \] where $\delta_n = s + 1$.

**Proof.** $C_n(K_t \times C_r, 0)$ and $C_n(K_t \times C_r, 1)$ follow from (1.8) and (1.9). For $2 \leq k \leq s-1$, we notice that there are $2t$ vertices each of distance $k$ from $(u_1, v_1)$, and there are $(2t(s-k) + t - 2)$ vertices each of distance more than $k$ from vertex $(u_1, v_1)$. Therefore, for $2 \leq k \leq s-1$,

\[ C_n((u_1, v_1), K_t \times C_r, k) = \sum_{j=0}^{n-1} \binom{2r}{j} \binom{\beta}{n-1-j}, \quad \beta = t(r - 2k + 1) - 2. \]

(See Fig. 2.1).

For $3 \leq n \leq 3t - 1$, then $\text{diam}_n K_t \times C_r \geq s$, and for $k = s$, there are $(2t - 1)$ vertices each of distance $s$ from $(u_1, v_1)$, and there are $(t - 1)$ vertices each of distance more than $s$ from $(u_1, v_1)$). Therefore,

\[ C_n((u_1, v_1), K_t \times C_r, s) = \sum_{j=0}^{n-1} \binom{3r - 2}{j} \binom{\tau - 1}{n-1-j}, \]

For $3 \leq n \leq t$, then $\text{diam}_n K_t \times C_r = s + 1$ by Proposition 2.1, and there are exactly $(t - 1)$ vertices each of distance $s + 1$ from vertex $(u_1, v_1)$, and there is no vertex of distance more than $s + 1$ from $(u_1, v_1)$. Thus,

\[ C_n((u_1, v_1), K_t \times C_r, s+1) = \binom{\tau - 1}{n-1}. \]

Since $K_t \times C_r$ is vertex-$n$-distance regular, then the proof of the theorem is completed. ■

3. Straight Hexagonal Chains

A **straight hexagonal chain** is a graph $\varsigma_t$ consisting of $t$ hexagons $H_1, H_2, \ldots, H_t$ such that $H_i$ and $H_{i+1}$, $1 \leq i \leq t - 1$, have one edge in common as shown in Fig. 3.1.
It is clear that \( p(\zeta_t) = 4t+2 \), \( q(\zeta_t) = 5t+1 \).

Let \( n \geq 2 \), and consider the vertex \( v_1 \). The \( n \)-diameter of \( \zeta_t \) is the \( n \)-distance of \((v_1,S)\) such that \( S \) is an \((n-1)\)-set consisting of vertices farthest from \( v_1 \). To find \( S \), we notice that

\[
\begin{align*}
d(v_1,v_{4t+2}) &= 2t+1, \\
d(v_1,v_{4t+1}) &= 2t, \\
d(v_1,v_{4t}) &= 2t-1, \\
d(v_1,v_{4t-1}) &= 2t-2, \\
d(v_1,v_{4t-2}) &= 2t-2,
\end{align*}
\]

in general

\[
d(v_1,v_i) = \left\lfloor \frac{i}{2} \right\rfloor,
\]

for \( i = 1, 2, 3, \ldots, 4t+2 \).

Therefore, if \( d_n(v_1,S) \) is maximum, then \( S \) consists of the first \( n-1 \) vertices from the sequence:

\[v_{4t+2}, v_{4t+1}, v_{4t}, v_{4t-1}, \ldots, v_5, v_4, v_3, v_2,\]

Thus, the vertex of \( S \) nearest to \( v_1 \) is \( v_{4t+n} \).

If \( n \) is even, then

\[
d(v_1,v_{4t+n}) = 2(t+1)-(n/2),
\]

and when \( n \) is odd,

\[
d(v_1,v_{4t+n}) = 2(t+1)-(n+1)/2.
\]

Therefore,

\[
d(v_1,v_{4t+n}) = 2(t+1)-\left\lfloor \frac{n}{2} \right\rfloor,
\]

which completes the proof of the following proposition.

**Proposition 3.1.** For \( t \geq 1 \), \( 2 \leq n \leq 4t+2 \),

\[
diam_n \zeta_t = 2(t+1)-\left\lfloor \frac{n}{2} \right\rfloor.
\]

To find the \( n \)- Wiener polynomial, \( n \geq 3 \), for \( \zeta_t \) we redraw \( \zeta_t \) as in Fig.3.2 with new labels for its vertices.

From Fig.3.2 we notice that \( \zeta_t \) is \( K_2 \times P_{2t+1} \) with the edges \( \{u_i'u_{i+1}: i=1,3,5,\ldots,2t-1\} \) removed.
Theorem 3.2. For \( t \geq 3, \) \( 3 \leq n \leq 4t+2, \) we have

\[
W_{n}(\varrho;x) = \sum_{k=0}^{\delta_n} C_{n}(\varrho,k)x^{k},
\]

where \( \delta_n \) is the \( n \)-diameter, and

\[
C_{n}(\varrho,0) = p \begin{pmatrix} p-1 \\ n-2 \end{pmatrix}, \quad p=4t+2,
\]

\[
C_{n}(\varrho,1) = p \begin{pmatrix} p-1 \\ n-1 \end{pmatrix} - (2t+4) \begin{pmatrix} p-3 \\ n-1 \end{pmatrix} - (2t-2) \begin{pmatrix} p-4 \\ n-1 \end{pmatrix},
\]

\[
C_{n}(\varrho,2) = 2 \{(t+2) \begin{pmatrix} p-3 \\ n-1 \end{pmatrix} + (t-1) \begin{pmatrix} p-4 \\ n-1 \end{pmatrix} - 2 \begin{pmatrix} p-5 \\ n-1 \end{pmatrix} - (2t-2) \begin{pmatrix} p-7 \\ n-1 \end{pmatrix}
- (t-1) \begin{pmatrix} p-8 \\ n-1 \end{pmatrix}\},
\]

\[
C_{n}(\varrho,3) = 2 \{2 \begin{pmatrix} p-5 \\ n-1 \end{pmatrix} + 2 \begin{pmatrix} p-6 \\ n-1 \end{pmatrix} + (t-4) \begin{pmatrix} p-7 \\ n-1 \end{pmatrix} + (t-1) \begin{pmatrix} p-8 \\ n-1 \end{pmatrix} - 2 \begin{pmatrix} p-9 \\ n-1 \end{pmatrix}
- 2 \begin{pmatrix} p-10 \\ n-1 \end{pmatrix} - (2t-5) \begin{pmatrix} p-12 \\ n-1 \end{pmatrix}\},
\]

for \( 4 \leq k \leq \delta_n, \)

\[
C_{n}(\varrho,k) = C_{n}(K_{2} \times P_{2t+1},k),
\]

in which \( C_{n}(K_{2} \times P_{2t+1},k) \) is given in Theorem 3.5.3. Ref[1].

Proof. \( C_{n}(\varrho,0) \) and \( C_{n}(\varrho,1) \) are obtained from (1.8) and (1.9).

To find \( C_{n}(\varrho,2) \) we notice that for \( u \in \{u_{1},u_{2},u_{2t+1},u_{2t+2}\} \) there are exactly 2 vertices of distance 2 from \( u \), and there are \( (p-5) \) vertices of distance more than 2 from \( u \). For this case, the number of pairs \((u,S)\) such that \( d_{n}(u,S)=2 \) is

\[
4 \sum_{j=1}^{p-5} \binom{2}{j} \begin{pmatrix} p-5 \\ n-1-j \end{pmatrix} = 4 \left[ \begin{pmatrix} p-3 \\ n-1 \end{pmatrix} - \begin{pmatrix} p-5 \\ n-1 \end{pmatrix} \right]. \quad \text{...(3.1)}
\]

If \( u \in \{u_{1},u_{2},u_{2t+1},u_{2t}\} \), then there are exactly 3 vertices of distance 2 from \( u \), and there are \( (p-6) \) vertices of distance more than 2 from \( u \). For these vertices, the number of pairs \((u,S)\) such that \( d_{n}(u,S)=2 \) is

\[
4 \sum_{j=1}^{p-6} \binom{3}{j} \begin{pmatrix} p-6 \\ n-1-j \end{pmatrix} = 4 \left[ \begin{pmatrix} p-3 \\ n-1 \end{pmatrix} - \begin{pmatrix} p-6 \\ n-1 \end{pmatrix} \right]. \quad \text{...(3.2)}
\]

If \( u \in \{u_{3},u_{4},\ldots,u_{2t-1},u_{2t}\} \), then there are 4 vertices of distance 2 from \( u \) and there are \( (p-8) \) vertices of distance more than 4 from \( u \). For these vertices \( u \), the number of pairs \((u,S)\) such that \( d_{n}(u,S)=2 \) is
If \( u \in \{ u_1, u_4, u_5, \ldots, u_{2t-3}, u_{2t-2} \} \), then there are 4 vertices of distance 2 from \( u \), and there are \( (p-7) \) vertices of distance more than 2 from \( u \). Therefore, the number of pairs \((u, S)\) such that \( d(u, v) = 2 \) for these vertices \( u \) is

\[
2(t-2) \sum_{j=1}^{n-1} \binom{4}{j} \binom{p-8}{n-1-j} = 2(t-2)\left[ \binom{p-3}{n-1} - \binom{p-7}{n-1} \right]. 
\]

...(3.4)

Summing the numbers in (3.1)-(3.4), we obtain the value of \( C_n(\varsigma, 2) \) as given in the theorem.

To find \( C_n(\varsigma, 3) \), we use the same method. If \( u \in \{ u_1, u_2, u_{2t+1}, u_{2t+2} \} \), then the number of \((u, S)\) pairs of n-distance 3 is

\[
4 \sum_{j=1}^{n-1} \binom{2}{j} \binom{p-7}{n-1-j} = 4\left[ \binom{p-5}{n-1} - \binom{p-7}{n-1} \right].
\]

...(3.5)

If \( u \in \{ u_1, u_2, u_{2t+1}, u_{2t+2} \} \), then the number of \((u, S)\) pairs is

\[
4 \sum_{j=1}^{n-1} \binom{3}{j} \binom{p-9}{n-1-j} = 4\left[ \binom{p-6}{n-1} - \binom{p-9}{n-1} \right].
\]

...(3.6)

If \( u \in \{ u_3, u_4, u_{2t+1}, u_{2t+2} \} \), then the number of \((u, S)\) pairs is

\[
4 \sum_{j=1}^{n-1} \binom{3}{j} \binom{p-11}{n-1-j} = 4\left[ \binom{p-8}{n-1} - \binom{p-11}{n-1} \right].
\]

...(3.7)

If \( u \in \{ u_3, u_4, \ldots, u_{2t+1}, u_{2t+2} \} \), then the number of \((u, S)\) pairs of n-distance 3 is

\[
2(t-2) \sum_{j=1}^{n-1} \binom{5}{j} \binom{p-12}{n-1-j} = 2(t-2)\left[ \binom{p-7}{n-1} - \binom{p-12}{n-1} \right].
\]

...(3.8)

If \( u \in \{ u_5, u_6, \ldots, u_{2t+1}, u_{2t+2} \} \), then the number of \((u, S)\) pairs of n-distance 3 is

\[
2(t-3) \sum_{j=1}^{n-1} \binom{4}{j} \binom{p-12}{n-1-j} = 2(t-3)\left[ \binom{p-8}{n-1} - \binom{p-12}{n-1} \right].
\]

...(3.9)

Summing the numbers in (3.5)-(3.9), we get \( C_n(\varsigma, 3) \) as given in the statement of the theorem.

From Fig.3.2, we notice that for \( 4 \leq k \leq \delta_n \), if \( d(u, v) = k \) in the graph \( \varsigma \) then it is also \( k \) in \( K_2 \times P_{2t+1} \), and conversely. Thus, if \( d(u, v) = k \) in \( \varsigma \) then it is also \( k \) in \( K_2 \times P_{2t+1} \), and conversely. Therefore,

\[
C_n(\varsigma, k) = C_n(K_2 \times P_{2t+1}, k), \text{ for } 4 \leq k \leq \delta_n.
\]

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