Wnil-Injective Modules

Raida D. Mahmood
Husam Q. Mohammad

College of Computer Sciences and Mathematics
University of Mosul

Received on:19/5/2009                 Accepted on:3/11/2009

ABSTRACT

A right module $M$ is called Wnil-injective if for any $0 \neq a \in N( R )$, there exists a positive integer $n$ such that $a^n \neq 0$ and any right $R$-homomorphism $f : a^n R \rightarrow M$ can be extended to $R \rightarrow M$. In this paper, we first give and develop various properties of right Wnil-injective rings, by which, many of the known results are extended. Also, we study the relations between such rings and reduced rings by adding some types of rings, such as SRB-rings, and other types of rings.

1. Introduction:

Throughout this paper $R$ is associative ring with identity, and $R$-module is unital. For $a \in R$, $r( a )$ and $l( a )$ denote the right annihilator and the left annihilator of $a$, respectively. We write $J( R )$, $Y( R )$, $Z( R )$, $N( R )$ and $\text{Soc}( R_R )$ for the Jacobson radical, the right (left) singular ideal, the set of nilpotent elements and right (left) socle of $R$, respectively.

( 1 ) A right $R$-module $M$ is called nil-injective [7] if for any $a \in N( R )$, any $R$-homomorphism $f : a R \rightarrow M$ can be extended to $R \rightarrow M$. Or equivalently, there exists $m \in M$ such that $f(x) = mx$ for all $x \in a R$. If $R_R$ is nil-injective, then we call $R$ is a right nil-injective ring. ( 2 ) A ring $R$ is said to be right NPP if $a R$ is projective for all $a \in N( R )$. ( 3 ) A ring $R$ is ( Von Neumann ) regular [4] provided that for every $a \in R$ there exists $b \in R$ such that $a = aba$. 
A ring \( R \) is called reduced if it contains no non-zero nilpotent elements. A ring \( R \) is called \( n \)-regular \([7]\) if \( a \in aRa \) for all \( a \in N(R) \).

Clearly, regular rings are \( n \)-regular, and reduced ring is right nil-injective. According to Cohn \([2]\), a ring \( R \) is called reversible if \( ab = 0 \) implies \( ba = 0 \) for \( a, b \in R \). A ring \( R \) is called strongly right bounded (briefly SRB) \([3]\) if every non-zero right ideal contains a non-zero two-sided ideal of \( R \).

A ring \( R \) is called right minsymmetric \([7]\) if \( kR \) minimal, \( k \in R \), implies that \( Rk \) is minimal.

2. \( Wnil \)-Injective Rings

This section is devoted to study \( Wnil \)-injective rings with some of their basic properties. Also we give a relation between such rings with \( n \)-regular rings, and reduced rings.

**Definition 2.1** \([7]\)

A right module \( M \) is called \( Wnil \)-injective if for any \( 0 \neq a \in N(R) \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and any right \( R \)-homomorphism \( f : a^nR \rightarrow M \) can be extended to \( R \rightarrow M \). Or equivalently, there exists \( m \in M \) such that \( f(x) = mx \) for all \( x \in a^nR \).

Clearly every right nil-injective module is right \( Wnil \)-injective. If \( R_R \) is \( Wnil \)-injective we call \( R \) is a right \( Wnil \)-injective ring.

**Lemma 2.2** \([7]\)

A ring \( R \) is right nil-injective if and only if \( lr(a) = Ra \) for all \( a \in N(R) \).

We start the section with the following theorem which extends Lemma 2.2.

**Theorem 2.3**

A ring \( R \) is a right \( Wnil \)-injective if and only if for any \( 0 \neq a \in N(R) \) there exists a positive integer \( n \) such that \( a^n \neq 0 \) and \( Ra^n = lr(a^n) \).

**Proof**

Suppose that a ring \( R \) is right \( Wnil \)-injective. Then for every \( 0 \neq a \in N(R) \), there exists a positive integer \( n \) such that \( a^n \neq 0 \) and any right \( R \)-homomorphism of \( a^nR \) into \( R \) extends to endomorphism of \( R_R \). It is clear \( Ra^n \subseteq lr(a^n) \). Let \( d \in lr(a^n) \), since \( r(a^n) = r(l(r(a^n))) \subseteq r(d) \). We may define a right \( R \)-homomorphism \( f : a^nR \rightarrow R \) by \( f(a^nb) = db \) for all \( b \in R \).

Since \( R_R \) is \( Wnil \)-injective, there exists \( y \in R \) such that \( f(a^n) = ya^n \). Then \( d = f(a^n) \in Ra^n \) which implies that \( lr(a^n) \subseteq Ra^n \) and so that \( lr(a^n) = Ra^n \).

Conversely, If \( c \in N(R) \), there exists a positive integer \( n \) such that
Wnil-Injective Modules

Let $f : c^n R \rightarrow R$ be any right $R$-homomorphism, then $r(c^n) \subseteq r(f(c^n))$ which implies $Rf(c^n) \subseteq lr(Rf(c^n)) = lr(Rc^n) = Rc^n$, and therefore $f(c^n) = de^n$ for some $d \in R$. This shows that $R$ is a right Wnil-injective ring. ■

**Lemma 2.4**: [1]

Let $R$ be a ring and $a \in R$. If $a^n - a^r a$ is regular for some positive integer $n$ and $r \in R$, then there exists $y \in R$ such that $a^n = a^r ya$. ■

**Lemma 2.5**: [4]

The following conditions are equivalent for a ring $R$

1. $R$ is a regular ring.
2. every principal right ideal of $R$ is generated by idempotent. ■

Next we prove the following result:

**Theorem 2.6**:

The following conditions are equivalent

1. $R$ is $n$-regular.
2. $N_1(R) = \{0 \neq x \in R \mid x^2 = 0\}$ is regular.
3. For any $a \in N(R)$, there exists a positive integer $n$ such that $a^n \neq 0$ and $a^n R$ is generated by an idempotent element.

**Proof**:

It is clear that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Let $a \in N(R)$ such that $a^{n+1} = 0$ and $a^n \neq 0$. Then $(a^n)^2 = 0$ so that $a^n \in N_1(R)$. This implies that $a^n$ is regular element of $R$ and so that there exists $b \in R$ such that $a^n = a^n ba$, which implies that $a^n R = eR$ by Lemma 2.5 and $a^n \neq 0$ for some $0 \neq e^2 = e \in R$.

(3) $\Rightarrow$ (1): Let $a \in N(R)$, then there exits integer $n$ such that $a^n R$ is generated by idempotent element. Hence $a^n = a^r ba$ for some $b \in R$. We shall show that $a$ is regular. In fact, if $n = 1$, then it holds. Let $n > 1$.

Put $d = a^{n-1} - a^{n-1} ba$, then $ad = 0$. If $d = 0$, then $a^{n-1} = a^{n-1} ba$. If $d \neq 0$ then $d^2 = (a^{n-1} - a^{n-1} ba) d = 0$. Since $N_1(R)$ is regular by (2), $d$ is regular. Hence $a^{n-1} = a^{n-1} ya$ for some $y \in R$ by Lemma 2.4. Therefore we always have $a^{n-1} = a^{n-1} z_1 a$ for some $z_1 \in R$. If $n-1 > 1$, then there exists $z_2 \in R$ such that $a^{n-2} = a^{n-2} z_2 a$ by the proceeding proof. Repeating the above-mentioned process, we get that there is $x \in R$ such that $a = axa$, i.e. $a$ is regular, which implies that $R$ is $n$-regular ring. ■

The following theorem characterizes $n$-regular in terms of a right Wnil-injective rings.
Theorem 2.7:

The following conditions are equivalent for a ring $R$.

(1) $R$ is $n$-regular ring.

(2) $R$ is a right Wnil-injective such that every cyclic singular right $R$-module is nil-injective.

Proof:

(1) $\implies$ (2): It is clear.

Assume (2). Let $b \in N(R)$. There exists a positive integer $n$ such that $b^n \neq 0$ and any right $R$-homomorphism of $b^n R$ into $R$ extends to endomorphism of $R_R$.

Then by Lemma 2.2 $Rb^n = lr(b^n) = lr(Rb^n)$. There exists a complement right ideal $L$ of $R$ such that $r(b^n) \oplus L$ is an essential right ideal. By hypothesis, the cyclic singular right $R$-module $R/(r(b^n) \oplus L)$ is nil-injective. Define a homomorphism $f : b^n R \to R/(r(b^n) \oplus L)$ by $f(b^n a) = a + (r(b^n) \oplus L)$ for all $a \in R$, then $f$ is well defined. Indeed if $b a_1 = b a_2$ then $a_1 - a_2 \in (r(b^n) \oplus L)$ so that $f(b^n a_1) = f(b^n a_2)$. There exists $z \in R$ such that $f(b^n a) = z b^n a + (r(b^n) \oplus L)$. So that $1 + (r(b^n) \oplus L) = f(b^n) = z b^n + (r(b^n) \oplus L)$ implies that $1 - z b^n = u + v$, where $u \in r(b^n), v \in L$.

For any $s \in r(Rb^n)$, $s = us + vs$ and $vs = s - us \in r(b^n) \cap L = 0$ which yields $v(r(Rb^n)) = 0$ which implies that $v \in l(r(Rb^n) = Rb^n)$. If $v = cb^n$, for some $c \in R$, then $1 - zb^n z = ub^n$ which implies that $b^n - b^n z b^n = b^n u + b^n c b^n$, and therefore $b^n = b^n (z + c) b^n$ [since $b^n u = 0$]. This proves that for any $0 \neq b \in N(R)$, there exists a positive integer $n$ such that $b^n \neq 0$ and $b^n R$ is generated by an idempotent element, so by Theorem 2.6 $R$ is an $n$-regular ring.

Corollary 2.8:

Let $R$ be a right Wnil-injective such that every cyclic singular right $R$-module is nil-injective, then $N(R) \cap J(R) = 0$

Proof:

If $a \in N(R) \cap J(R)$, then $a = aba, b \in R$, by Theorem 2.7. Hence $a(1 - ba) = 0$, because $a \in J(R)$. Hence $1 - ba$ is invertable and so $a = 0$. Therefore $N(R) \cap J(R) = 0$.

Theorem 2.9:

The following conditions are equivalent for a ring $R$.

(1) $R$ is a $n$-regular ring $R$.

(2) $R$ is a right Wnil-injective a right NPP-ring.
**Proof:**

(1) ⇒ (2): It is clear.

(2) ⇒ (1): Suppose that \(0 \neq a \in N(R)\), then by Lemma 2.2 there exists a positive integer \(n\) such that \(a^n \neq 0\) and \(Ra^n = lr( a^n )\). Since \(R\) is a right NPP-ring, \(r( a^n ) = (1 - e)R,\) \(0 \neq e^2 = ee \in R\). Therefore \(Ra^n = Re\) which implies that \(a^nR = gR\) for some \(0 \neq g^2 = g \in R\). So that, by Theorem 2.6, \(R\) is an \(n\)-regular ring. ■

3. Ring Whose Simple Singular \(R\)-module are Wnil-Injective.

In this section, we give an investigation of several properties for rings whose simple singular right \(R\)-modules are Wnil-injective due to J. Ch. Wei and J. H. Chen [7]. Also we study the relations between such rings, reduced rings, by adding some types of rings such as SRB-ring, and other types of rings.

**Lemma 3.1:** [5]

If \(R\) is a semi-prime and SRB-ring, then \(R\) is reduced. ■

We begin with the following theorem.

**Theorem 3.2:**

Let \(R\) be an SRB-ring whose every simple singular right \(R\)-module is Wnil-injective. Then \(R\) is a reduced ring.

**Proof:**

By Lemma 3.1, it is enough to show that \(R\) is semi-prime. Suppose that there exists a non-zero right ideal \(U\) of \(R\) such that \(U^2 = 0\). Then there exists a non-zero elements \(a \in U\) such that \(a^2 = 0\). First observe that \(r( a )\) is an essential right ideal of \(R\). If not, there exists a non-zero right ideal \(L\) of \(R\) such that \(r( a ) \oplus L\) is right essential in \(R\). Since \(R\) is SRB, there is a non-zero ideal \(I\) of \(R\) such that \(I \subseteq L\). Now \(aI \subseteq aR \cap I \subseteq r( a ) \cap L = 0\). Hence \(I \subseteq r( a ) \cap L = 0\). This is a contradiction. Thus \(r( a )\) must be proper essential right ideal of \(R\). Hence there exists a maximal right ideal \(M\) of \(R\) containing \(r( a )\). Clearly \(M\) is an essential right ideal of \(R\), and \(R / M\) is Wnil-injective. So any \(R\)-homomorphism of \(aR\) into \(R / M\) extends to one of \(R\) into \(R / M\). Let \(f: aR \rightarrow R / M\) be defined by \(f( ar ) = r + M\). Then \(f\) is well-defined \(R\)-homomorphism. Since \(R / M\) is Wnil-injective, so there exists \(c \in R\) such that \(1 + M = f( a ) = ca + M\) which implies that \(1 - ca \subseteq M\). Now \(aca \in aR \subseteq U^2 = 0\), hence \(ca \in r( a ) \subseteq M\), and so \(1 \in M\), which is a contradiction. Therefore \(R\) must be semi-prime, and hence \(R\) is a reduced ring. ■

Following [6], a ring \(R\) is called a right **DS-ring** if every minimal right ideal of \(R\) is a direct summand.
Proposition 3.3:
Let \( R \) be a ring whose every simple right \( R \)-module is W-nil-injective. Then

1. \( J( R ) \cap \text{Soc}( R_R ) = 0 \)
2. \( J( R ) \) is a reduced ideal of \( R \).

Proof:

(1) If \( J( R ) \cap \text{Soc}( R_R ) \neq 0 \), then there exists a minimal right ideal \( kR \) of \( R \) with \( kR \subseteq J( R ) \). If \( kR \) is a direct summand then \( kR = eR \) for some \( 0 \neq e^2 = e \in R \) and we get \( e \in J( R ) \) which is a contradiction. So that \( (kR)^2 = 0 \). Since \( r( k ) \) is a maximal right ideal of \( R \), then \( R / r( k ) \) is W-nil-injective. Set \( f : kR \to R / r( k ) \) defined by right \( R \)-homomorphism. Since \( R / r( k ) \) is right W-nil-injective ring and \( k \in N( R ) \), there exists \( c \in R \) such that \( f( kr ) = ckr + r( k ) \). Therefore \( 1 + r( k ) = f( k ) = ck + r( k ) \) which implies \( 1 - ck \in r( k ) \). Since \( k \in J( R ) \), then \( ck \in J( R ) \subseteq r( k ) \) which implies \( 1 \in r( k ) \), which is also a contradiction. Therefore \( J( R ) \cap \text{Soc}( R_R ) = 0 \).

(2) Let \( 0 \neq x \in J( R ) \) such that \( x^2 = 0 \). Since \( x \neq 0 \), then \( r( x ) \subseteq M \) for some maximal right ideal \( M \) of \( R \). Define a right \( R \)-homomorphism \( f : xR \to R / M \) such that \( f( xr ) = r + M \) for all \( r \in R \). Then \( f \) is well-defined right \( R \)-homomorphism. Since \( R / M \) is a right W-nil-injective ring and \( x \in N( R ) \), there exists \( c \in R \) such that \( f( xr ) = cxr + M \). Therefore

\[
1 + M = f( x ) = cx + M,
\]

which implies that \( 1 - cx \in M \), and so \( 1 \in M \), which is a contradiction. Hence \( J( R ) \) is reduced.

Lemma 3.4:[6]
Let \( R \) be a ring. Then \( R \) is a right DS-ring if and only if \( J( R ) \cap \text{Soc}( R_R ) = 0 \).

Corollary 3.5:
Let \( R \) be a ring whose every simple right \( R \)-module is W-nil-injective. Then \( R \) is a right DS-ring.

Following [6] a ring \( R \) is called right MC2-ring if \( eRa=0 \) implies \( aRe=0 \), where \( a, e^2 = e \in R \) and \( eR \) is minimal right ideal of \( R \). Or equivalently, if \( K = eR \) are minimal, \( e^2 = e \in R \); then \( K = gR \) for some \( g^2 = g \in R \).

Lemma 3.6:[7]
Suppose that every simple singular right \( R \)-module is W-nil-injective. Then \( R \) is a right MC2 ring if and only if \( R \) is a right minsymmetric ring.
Theorem 3.7:

Suppose that every simple singular right R-module is Wnil-injective. Then $\text{Soc}( R_R ) \cap J( R ) = \text{Soc}( R_R ) \cap Y( R )$ if and only if $R$ is a right minsymmetric ring.

Proof:

It is clear that $\text{Soc}( R_R ) \cap Y( R ) \subseteq \text{Soc}( R_R ) \cap J( R )$. Now, to prove $\text{Soc}( R_R ) \cap J( R ) \subseteq \text{Soc}( R_R ) \cap Y( R )$, let $kR$ be a minimal right ideal of $R$, and $kR \subseteq \text{Soc}( R_R ) \cap J( R )$. Since $r( k )$ is a maximal right ideal of $R$, and if $r( k )$ is not essential of $R$, then $r( k )$ is a direct summand of $R$. So $kR \cong R / r( k ) \cong eR$, $e^2 = e \in R$. Hence $kR$ is a direct summand of $R$. Since $R$ is a right minsymmetric ring, then by Lemma 3.6, $R$ is a right MC2. Then $kR = gR$, $g^2 = g \in R$ so that $g \in J( R )$, which is a contradiction. Hence $r( k )$ is essential in $R$ and we have $kR \subseteq Y( R )$. Therefore $kR \subseteq \text{Soc}( R_R ) \cap Y( R )$.

Conversely, Assume that $\text{Soc}( R_R ) \cap J( R ) = \text{Soc}( R_R ) \cap Y( R )$, if $kR$ is a minimal right ideal of $R$, and $kR = eR$, $e^2 = e \in R$. If $kR \subseteq J( R )$, then $kR \subseteq \text{Soc}( R_R ) \cap J( R ) = \text{Soc}( R_R ) \cap Y( R )$. Hence $kR$ and $eR$ are singular right ideals, a contradiction. Then $kR \not\subseteq J( R )$, and we get $(kR)^2 \neq 0$, so $(kR)^2 = kR = gR$, $g^2 = g \in R$. Therefore $R$ is a right minsymmetric ring.

From Proposition 3.3 and Theorem 3.7 we conclude following corollary.

Corollary 3.8:

Let $R$ be a right minsymmetric ring and every simple singular right $R$-module is Wnil-injective. Then $\text{Soc}( R_R ) \cap Y( R ) = 0$.

Theorem 3.9:

Let $R$ be reversible and $N( R )$ is an ideal of $R$. Then, the following conditions are equivalent:

1. $R$ is reduced
2. $R$ is n-regular
3. $R$ is right nil-injective and NPP.
4. $R$ is right Wnil-injective and NPP.
5. every simple right $R$-module is Wnil-injective.
6. every simple singular right $R$-module is Wnil-injective.
Proof:
It is obvious that (1) ⇔ (2) ⇔ (3) ⇒ (4) ⇒ (5) ⇒ (6).

(6) ⇒ (1): Let \( a \in R \) with \( a^2 = 0 \). If \( a \neq 0 \), then \( r(a) \neq R \), so there exists a right ideal \( L \) of \( R \) such that \( r(a) \oplus L \) is essential in \( R \). If \( r(a) \oplus L \neq R \), there exists a maximal right ideal \( M \) of \( R \) containing \( r(a) \oplus L \). Clearly, \( M \) is an essential right ideal of \( R \), and by hypothesis, \( R / M \) is Wnil-injective. So there exists \( c \in R \) such that \( 1-ca \in M \). Since \( N(R) \) is an ideal of \( R \), \( ca \in N(R) \), so \( 1-ca \) is invertible. Hence \( M=R \), which is a contradiction. This show that \( r(a) \oplus L = R \).

Let \( r(a) = eR \), \( e^2 = e \in R \). Clearly, \( a = ae = ea = 0 \), which is a contradiction. So \( a = 0 \). ■
REFERENCES


