Maximal Generalization of Pure Ideals

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ABSTRACT

The purpose of this paper is to study the class of the rings for which every maximal right ideal is left GP-ideal. Such rings are called MGP-rings and give some of their basic properties as well as the relation between MGP-rings, strongly regular ring, weakly regular ring and kasch ring.

1- Introduction:

Throughout this paper, R denotes as associative ring with identity. An ideal I of a ring R is said to be right(left) pure if for every a∈I, there exists b∈I such that a=ab (a=ba). This concept was introduced by Fieldhouse [6], [7], Al-Ezeh [2],[3] and Mahmood [9].

Recall that:-
1- A ring R is regular if for every a∈R there exists b∈R such that a=aba, if a=a^2 b, R is called strongly regular.
2- A ring without non-zero nilpotent elements is called reduced.
3- For any element a∈R, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively.
4- A ring R is said to be a left(right) uniform ring if and only if every non-zero left(right) ideals is essential.
5- Following [10], a ring R is said to be semi commutative if xy=0 implies that xRy=0, x,y∈R. Clearly every reduced ring is semi commutative. It is easy to see that R is semi commutative if and only if every left(right) annihilator in R is a two-sided ideal.
6- Y(R), J(R) are respectively the right singular ideal and the Jacobson radical of R.
2- MGP-rings

In this section, the concept of maximal GP-ideals is introduced and we use it to define MGP-rings. We study such rings and give some of their basic properties.

Following [8], an ideal I of a ring R is said to be right (left) GP-ideal (generalized pure ideal), if for every a in I, there exists b in I and a positive integer n such that $a^n = a^n b$ ($a^n = b a^n$).

Definition 2.1 :

A ring R is called a right (left) MGP-ring if and only if every maximal right (left) ideal is left (right) GP-ideal.

Example:

Let $\mathbb{Z}_{12}$ be the ring of the integers modulo 12.

Then the maximal ideals, $I = \{0, 3, 6, 9\}$, $J = \{0, 2, 4, 6, 8, 10\}$ are GP-ideals.

The following theorem gives some interesting characteristic properties of right MGP-rings. Before that we need the next lemma in our proof.

Lemma 2.2:

Let $a$ be a non zero element of a ring $R$ and let $l(a) = 0$. Then for every positive integer $n$, $l(a^n) = 0$.

Proof: obvious #

Theorem 2.3 :

If $R$ is a right MGP-ring and every ideal is principal, then any left regular element is right invertible.

Proof :

Let $0 \neq c \in R$, such that $l(c) = 0$. If $c R \neq R$, then there exists a maximal right ideal $M$ containing $c R$. Since $R$ is right MGP-ring, then $M$ is a left GP-ideal, there exists $d \in M$ and a positive integer $n$, such that $c^n = dc^n$ and $d = cx$, for some $x \in R$.

So $(1-cx) \in l(c^n)$, Since $l(c) = 0$, then by Lemma (2.2) we have $l(c^n) = 0$, thus $cx = 1 \in M$, this contradicts $c R \neq R$. Therefore $c R = R$, and hence $c$ is a right invertible. #

Lemma 2.4 :

Let $R$ be a reduced ring. Then for every $a \in R$, and every positive integer $n$, $a^n R \cap r(a^n) = 0$.

Proof: See [8]
Proposition 2.5:
Let $R$ be a reduced, MGP-ring. Then for every $a$ in $R$ and a positive integer $n$, $r\left(a^n\right)$ is a direct summand of $R$.

Proof:
To prove $r\left(a^n\right)$ is a direct summand, we claim that $a^n R + r\left(a^n\right) = R$. If this is not true, let $M$ be a maximal right ideal containing $a^n R + r\left(a^n\right)$. Since $R$ is MGP-ring, so $(a^n)^m = b (a^n)^m$ for some $b \in M$ and a positive integer $m$. This implies $(1-b) \in l( a^{mn} ) = r( a^n ) \subseteq M$ (R is reduced), and so $1 \in M$, a contradiction. Hence $a^n R + r\left(a^n\right) = R$.

Now, since $a^n R \cap r\left(a^n\right) = 0$, then $r\left(a^n\right)$ is a direct summand.

Recall that, a ring $R$ is called a right (left) MP-ring if every maximal right (left) ideal is a left (right) pure.

We consider the condition (*): $R$ satisfies $l( b^n ) \subseteq r(b)$ for any $b \in R$ and a positive integer $n$.

Theorem 2.6:
Let $R$ be a ring satisfying (*). Then $R$ is a right MGP-ring if and only if $R$ is strongly regular.

Proof:
If this is not true let $R$ be a right MGP-ring and let $b$ be any element in $R$. We shall prove that $bR + r\left(b\right) = R$.

If this is not true let $M$ be a maximal right ideal containing $bR + r\left(b\right)$. Since $R$ is an MGP-ring, then there exists $a \in M$ and a positive integer $n$ such that $b^n = ab^n$ which implies that $(1-a) \in l( b^n ) \subseteq r(b) \subseteq M$, thus $1 \in M$, a contradiction. Therefore $bR + r\left(b\right) = R$.

In particular, $b u + v = 1$, for some $u \in R$, $v \in r\left(b\right)$. So $b = b^2 u$, therefore $R$ is strongly regular.

Conversely; assume that $R$ is strongly regular, then by [1], $R$ is regular and reduced. Also by [9], $R$ is an MP-ring and semi commutative, then $R$ is an MGP-ring.

Proposition 2.7:
Let $R$ be a right MGP-ring satisfying (*). Then $Y(R) = 0$.

Proof:
If $Y(R) \neq 0$, then by a Lemma (7) of [10]; there exists $0 \neq a \in Y\left(R\right)$ with $a^2 = 0$. From Theorem (2.6) $R$ is strongly regular, that is $a = a^2 b$, for some $b \in R$. Hence $a = 0$, contradiction. Therefore $Y(R) = 0$. #
Proposition 2.8:
If R is a right MGP – ring, then any reduced principal right ideal of R is a direct summand.
Proof: Let I = aR be a reduced principal right ideal of R. If aR + r(a) ≠ R, then there exists a maximal right ideal M of R containing aR + r(a).

Now, since R is a right MGP-ring and a ∈ M, then there exists b ∈ M and a positive integer n such that a^n = b a^n, and hence (1-b)a^n = 0. Since I is reduced then we have (1-b) ∈ l(a^n) = r(a^n) = r(a) ⊆ M, this implies that 1 ∈ M, which contradicts M ≠ R. Therefore, aR + r(a) = R, thus a = a^2 c for some c ∈ R. If we set d = a^2 ∈ I, then a = a^2 d implies that a = ada and hence aR = e R, where e = ad is an idempotent element. Then by [6], aR is a direct summand. #

Proposition 2.9:
Let R be a right MGP-ring satisfying (*). If a^n b = 0, for any a, b ∈ R and a positive integer n, then r(a^n) + r(b) = R.
Proof: Assume that r(a^n) + r(b) ≠ R. Let M be a maximal right ideal containing r(a^n) + r(b). Since R is a right MGP-ring and a^n b = 0 implies that b ∈ r(a^n) ⊆ M, there exists c ∈ M and a positive integer m such that b^m = cb^m, so (1 - c) ∈ l(b^m) ⊆ r(b) ⊆ M, which implies that 1 ∈ M, which is a contradiction. Therefore r(a^n) + r(b) = R.

Theorem 2.10:
Let R be a uniform semi commutative, MGP-ring and every ideal is principal. Then R is a division ring.
Proof: Let 0 ≠ a ∈ R and aR ≠ R, and let M be a maximal right ideal containing aR. Since R is an MGP-ring, then there exists b ∈ aR ⊆ M, and a positive integer n such that a^n = ba^n.
This implies that a^n = aca^n, for some c ∈ R. Since R is uniform so every ideal is an essential ideal.

Let x ∈ r(ar) ∩ a^n R. Then acx = 0 and x = a^n z for some z ∈ R, so aca^n z = 0, yields a^n z = 0 = x. Therefore, r(ac) ∩ a^n R = 0, since R is a uniform ring and a^n R ≠ 0, then r(ac) = 0. Since R is semi commutative, I(ac) = 0, then by Theorem (2.3) ac is a right invertible element, so there exists v ∈ R such that acv = 1. Hence a(cv) = 1 ∈ M, which is a contradiction. Therefore aR = R.

Now, since ar = 1 (aR = R), we have ara = a which implies that (1-ra) ∈ r(a) ⊆ l(a) ⊆ l(ar) = r(ar) = 0. Therefore, (1-ra) = 0, whence ra = 1, so a is a left invertible. Thus R is a division ring. #
3-The relation between MGP-rings and other rings

In this section we give further properties of the MGP-rings and link between MGP-rings and other rings.

We shall begin this section with the following result, which gives the connection between MGP-rings and weakly regular rings.

Following [11], a ring $R$ is a right (left) weakly regular if $I^2 = I$ for each right (left) ideal $I$ of $R$. Equivalently, if $a \in aRa$ ($a \in RaRa$) for every $a$ in $R$. Then $R$ is called weakly regular.

**Theorem 3.1:**

Let $R$ be a right MGP-ring and satisfying (*). Then $R$ is a reduced weakly regular ring.

**Proof:** Let $a$ be a non-zero element in $R$ with $a^2 = 0$. Let $M$ be a maximal right ideal containing $r(a)$. Since $a \in r(a) \subseteq M$ and $R$ is an MGP-ring, then there exists $b \in M$ and a positive integer $n$ such that $a^n = ba^n$, which implies that $(1 - b) \in l(an) \subseteq r(a) \subseteq M$, yielding $1 \in M$, which is a contradiction.

Therefore, $a = 0$, and hence $R$ is a reduced ring. We show that $RxR + r(x) = R$, for any $x \in R$.

Suppose that there exists $y \in R$ such that $RyR + r(y) \neq R$.

Then there exists a maximal right ideal $M$ of $R$ containing $RyR + r(y)$. Since $R$ is a right MGP-ring, there exists $a \in M$ and a positive integer $n$ such that $y^n = a y^n$ implying that $(1 - a) \in l(y^n) \subseteq r(y) \subseteq M$, whence $(1 - a) \in M$ and so $1 \in M$ implies that $M = R$, which is a contradiction.

Therefore, $RxR + r(x) = R$, for any $x \in R$.

Hence $R$ is a right weakly regular ring. Since $R$ is reduced, it also can be easily verified that $R$ is a weakly regular ring. #

**Definition 3.2:** [9]

A ring $R$ is said to be a right (left) Kasch ring if every maximal right (left) ideal is a right (left) annihilator.

**Theorem 3.3:**

Every semi-commutative right MGP-ring is a right Kasch ring.

**Proof:** Let $M$ be any maximal right ideal of $R$ and let $Y(R)$ be the right singular ideal of $R$.

If $M \cap Y(R) = 0$, then for any $y \in Y(R)$, $y \not\in M$, this implies that $r(y)$ is an essential right ideal of $R$.

Let $x \in r(y) \cap r(1-y)$, then $yx = 0$ and $(1-y)x = 0$ yields $x = yx = 0$. 

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Therefore \( r(y) \cap r(I-y) = 0 \), whence \( r(1-y) = 0 \). Since \( R \) is semi commutative ring, then we have \( I(1-y) = 0 \).

By Theorem (2.3), \((1-y)\) is an invertible element of \( R \). Hence \( y \in J \subseteq M \), a contradiction.

Thus \( M \cap Y(R) \neq \emptyset \). Let \( 0 \neq a \in M \cap Y(R) \).

Since \( R \) is an MGP-ring, then there exists \( b \in M \) and a positive integer \( n \) such that \( a^n = ba^n = ara^n \). We claim that \( r(ar) \cap a^n R = 0 \).

If not, let \( d \in r(ar) \cap a^n R \). Then \( ard = 0 \) and \( d = a^n x \) for some \( x \in R \), so \( ara^n x = 0 \) implies that \( a^n x = 0 = d \). Therefore, \( r(ar) \cap a^n R = 0 \). But \( r(ar) \) is essential, then \( a^n R = 0 \) and hence \( a^n x = 0 \), for all \( x \in R \) implies that \( a^n \in I(x) = r(x) \). Therefore, \( M = r(x) \). Thus \( R \) is a right Kasch ring. #

**Corollary 3.4:**

Let \( R \) be a reduced MGP-ring. Then \( R \) is a Kasch ring.

**Proof:** Since \( R \) is a reduced right MGP-ring. Then by Theorem (3.3) \( R \) is a Kasch ring. #
REFERENCES


