On ERT And MERT-Rings

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ABSTRACT

The main purpose of this paper is to study ERT and MERT rings, in order to study the connection between such rings and II-regular rings.
1- Introduction:
Throughout this paper, $R$ denotes an associative ring with identity, and all modules are unitary right $R$-module. Recall that; 1- An ideal $I$ of the ring $R$ is essential if $I$ has a non-zero intersection with every non-zero ideal of $R$; 2- A ring $R$ is said to be $\prod$-regular if for every $a$ in $R$ there exist a positive integer $n$ and $b$ in $R$ such that $a^n = a^n b a^n$ 3- A right $R$-module $M$ is said to be GP-injective if, for any $0 \neq a \in R$, there exists a positive integer $n$ such that $a^n \neq 0$ and any right $R$-homomorphism of $a^n R$ into $M$ extends to one of $R$ into $M$. 4- For any element $a$ in $R$, $r(a), I(a)$ denote the right annihilator of $a$ and the left annihilator of $a$, respectively.

2- ERT-R1NGS:
Following [3J, a ring $R$ is said to be ERT-ring if every essential right ideal of $R$ is a two-sided ideal.

Definition 2-1:
A ring $R$ is said to be right weakly regular if for all $a$ in $R$, there exists $b$ in $RaR$ such that $a = ab$, or equivalently every right ideal of $R$ is idempotent.

We begin this section with the following main result:

Theorem 2.2:
If $R$ is ERT-ring with every essential right ideal is idempotent, then $R$ is weakly regular.

Proof:
For any $a \in R$, if $RaR$ not essential, then there exists an ideal $I$ such that $K = RaR \oplus I$ is essential then $K = K^2$.
In order to prove that $K$ is weakly regular, we need to prove $RaR = (RaR)^2$.
For a $a \in K$, we have $a \in K^2$, that is $a \in (RaR \oplus I)^2$
Thus $a = (rar + i)(sas' + i')$ for some $r, r', s, s' \in R$ and $i, i' \in I$.
This implies that $a = (rar + i)sas' + (rar + i) i'$
\[ = rar'as' + isas' + (rar' + i) i' \]
but \( isas' \in I \cap RaR = 0 \), also we have \( (rar' + i)i' \in RaR \cap I = 0 \).
Therefore \( a = (rar')(isas') \in (RaR)^2 \), this implies that \( RaR \subseteq (RaR)^2 \) Thus \( RaR = (RaR)^2 \), this proves that \( R \) is weakly regular ring.

Following [2], the singular submodule of \( R \) is \( Y(R) = \{ y \in R, r(y) \) is essential right ideal of \( R \} \).

**Theorem 2.3:**
Let \( R \) be a semi-prime ERT right GP-injective ring. Then \( R \) is a right non singular.

**Proof:**
Let \( E \) be an essential right ideal of \( R \). Then \( E \) is a two-sided ideal, and hence \( l(E) \) is a two-sided ideal of \( R \).
Now \( (l(E) \cap E)^2 \subseteq (E)E = 0 \).
Since \( R \) is semi-prime, then \( l(E) \cap E = 0 \), whence \( l(E) = 0 \). This proves that \( R \) is right non singular.

**3- MERT-RINGS:**
Following [3], a ring \( R \) is said to be MERT-ring if every maximal essential right ideal of \( R \) is a two-sided ideal.

**Theorem 3.1:**
Let \( R \) be an MERT-ring, if for any maximal right ideal \( A/\) of \( R \), and for any \( b \in M \), \( bR/bM \) is GP-injective, then \( R \) is strongly Pi-regular ring.

**Proof:**
Let \( b \) be a non-zero element in \( R \), we claim that \( b^n r + r(b^n) = R \).
If \( b^n r + r(b^n) \neq R \), let \( M \) be a maximal right ideal containing \( b^n r + r(b^n) \). Then \( M \) is essential right ideal of \( R \).
If \( bR = bM \), then \( b = bc \), for some \( c \) in \( M \), this implies \( (1-c) \in r(b) \subseteq r(b^n) \subseteq M \), therefore \( 1 \in M \), this contradics \( M \neq R \).
Now, since \( R/ M \cong bR/bM \). Then \( R/ M \) is GP-injective.

Now, define \( f: b^n R \to R/ M \) by \( f(b^n r) = r + M \), note that \( f \) is a well-defined \( R \)-homomorphism.

Since \( R/M \) is GP-injective, then there exists \( c \in R \), such that:

\[
I+M = f(b^n) = cb^n + M
\]

and so \((1-cb^n) \in M\), since \( b^n \in M\), and \( R \) is MERT-ring, this implies that \( M \) is a two-sided ideal, and hence \( c b^n \in M \).

Thus \( I \subseteq M \), a contradiction.

Therefore \( b^n R + r(b^n) = R \).

In particular \( l = b^n u + v; v \in r(b^n) \), \( u \in R \).

Thus \( b^n = b^{2n} u \) and therefore \( R \) is strongly \( \Pi \)-regular ring.

**Theorem 3.2:**

If \( R \) is MERT-ring with every simple singular right ideal is GP-injective, then \( Y(R) = 0 \).

**Proof:**

If \( Y(R) \neq 0 \), by Lemma (7) of [6], there exists \( 0 \neq y \in Y(R) \) with \( y^2 = 0 \). Let \( L \) be a maximal right ideal of \( R \), set \( L = y R + r(y) \), we claim that \( L \) is essential right ideal of \( R \). Suppose this is not true, then there exists a non-zero ideal \( T \) of \( R \) such that \( L \cap T = (0) \). Then \( yRT \subseteq LT \subseteq L \cap T = 0 \) implies \( T \subseteq r(y) \subseteq L \), so \( L \cap T = (0) \). This contradiction proves that \( L \) is an essential right ideal, that is \( R/L \) is simple singular and hence \( R/L \) is GP-injective.

Now: Let \( f; yR \to R/L \) be defined by \( f(yr) = r + L \), then \( f \) is a well-defined \( R \)-homomorphism.

Since \( R/L \) is GP-injective, so \( \exists c \in R \), such that \( l + L = f(y) = cy + L \).

Hence \( l + L = cy + L \), implies that \( l - cy \in L \).

Since \( R \) is MERT, then \( ey \in L \) and thus \( l \in L \), a contradiction.

Therefore \( Y(R) = [0] \).

Following [1], a ring \( R \) is zero insertive (briefly ZI) if for \( a, b \in R \), \( ab = 0 \) implies \( aRb = 0 \).
Theorem 3.3:
Let $R$ be a ZT ring. If every simple singular right-modules is GP-injective which is left self-injective, then $R$ is strongly $H$-regular ring.

Proof:
Since $R$ is simple singular GP-injective, then $R$ is semi-prime, by Lemma (4) of [5]. Thus for any left ideal $I$, $L(I) \cap l = 0$.

Since $R$ is simple singular GP-injective and $ZI$, then $R$ is reduced and hence $r(a) = l(a)$ for any element $a$ in $R$.

Thus $l(r(a)) \cap l(a) = l(l(a)) \cap l(a) = 0$.

Since $R$ is left self-injective ring, then $aR$ is a right annihilator, by Proposition (4) of [4].

Since $r(a) \subseteq r(a^n)$, then $a^nR = r(a^n)$.

Now, since $R = r(l(r(a))) + r(l(l(a)))$ then we have $R = r(l(r(a^n))) + r(l((a^n))) = r(a^n) + a^nR$.

In particular, for some $b$ in $R$, and $d$ in $r(a^n)$.

Thus $a^n = a^n b$.

Therefore $R$ is strongly $\Pi$-regular.
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