An Approximate Solution to The Newell-Whitehead Equation by
Adomian Decomposition Method

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ABSTRACT

In this paper, we solved the Newell-Whitehead equation approximately using Adomain Decomposition method and we have compared this solution with the exact solution; we found that the solution of this method is so close to the exact solution and this solution is slower to converge to the exact solution when we increase \(t\) however, this method is effective for this kind of problems.


1. Introduction

The decomposition method was first introduced by Adomian since the beginning of the 1980s [7]. The Adomian method [3] can be used for solving a wide range of problems whose mathematical models yield equation or system of equations involving algebraic, differential, integral and integro-differential. In this method the solution is considered as the sum of an infinite series, rapidly converging to an accurate solution.

It is well known that the key of the method is to decompose the nonlinear term in the equations into a peculiar series of polynomials \( \sum_{n=1}^{\infty} A_n \), where \( A_n \) are the so-called Adomian polynomials [4–6].

This iterative method has been proven to be rather successful in dealing with linear problems as well as nonlinear. Adomian gives the solution as an infinite series usually converging to an accurate solution.

An Analytical Solution of the Stochastic Navier–Stokes System is shown by Adomian [3]. The decomposition method used to solve a system of partial differential equations and in reaction–diffusion to the Brusselator model and finding that the Adomian series solution gives an excellent approximation to the exact solution [20]. Wazwaz [22] developed a fast and accurate algorithm for the solution of sixth-order boundary value problems (BVP) and the modified decomposition method [19, 21]. Abbasbandy [1, 2] and Allan [8] studied some efficient numerical algorithms to solve a system of two nonlinear equations (with two variables) based on Newton’s method and
a numerical solution of the Blasius equation. Wang [18] presented a new algorithm for solving the classical Blasius equation.

Hashim [11] studied the Adomian decomposition method for solving BVPs for fourth-order integro-differential equations showing that with a few modifications the Adomian’s method can be used to obtain the known results of the special functions of mathematical physics and the Blasius equation [13].

Kechil et al. [15] applied a non-perturbative solution of free-convective boundary-layer equation by ADM. Chang [9] presented a decomposition solution for fins with temperature dependent surface heat flux. Chiu and Chen [10] used a decomposition method for solving the convective longitudinal fins with variable thermal conductivity. ADM also have used by several researchers to solve a wide range of physical problems in various engineering fields such as fluid flow and porous media simulation [6, 12, 14, 17].

2. The Principle of the Adomian Decomposition Method (ADM) [3]

Beginning with an equation

\[ Fu(t) = g(t) \]  

where \( F \) represents a general nonlinear ordinary differential operator involving both linear and nonlinear terms. The linear term is decomposed into \( L + R \), where \( L \) is easily invertible and \( R \) is the remainder of the linear operator. For convenience, \( L \) may be taken as the highest order derivative which avoids difficult integrations which result when complicated Green’s functions are involved. Thus the equation (1) can be written as

\[ Lu + Ru + Nu = g \]  

where \( Nu \) represents the nonlinear terms. Solving for \( Lu \),

\[ Lu = g - Ru - Nu \]  

Because \( L \) is invertible, operating with its inverse \( L^{-1} \) yields

\[ L^{-1} Lu = L^{-1} g - L^{-1} Ru - L^{-1} Nu \]  

An equivalent expression is

\[ u = \Phi + L^{-1} g - L^{-1} Ru - L^{-1} Nu \]  

where \( \Phi \) is the integration constant and satisfies \( L \Phi = 0 \). For initial-value problems we conveniently \( L^{-1} \) for \( L = \frac{d^n}{dt^n} \) as the n-fold definite integration operator from \( t_0 \) to \( t \).

For the operator \( L = \frac{a^n}{dt^n} \) for example, we have \( L^{-1} Lu = u-u(0)-tu'(0) \) and therefore

\[ u = u(0) + tu'(0) + L^{-1} g - L^{-1} Ru - L^{-1} Nu \]  

For boundary value problems, indefinite integrations are used and the constants are evaluated from the given conditions. Solving for \( u \) yields

\[ u = A + Bt + L^{-1} g - L^{-1} Ru - L^{-1} Nu \]  

The Adomian decomposition method [22] assumes an infinite series solution for unknown function \( u \) given by
\[
\sum_{n=0}^{\infty} u_n = u
\]  

(8)

and the nonlinear term \( Nu \), assumed to be analytic function \( f(u) \), is decomposed as follows:

\[
Nu = f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ..., u_n)
\]

(9)

where \( A_n \) are the appropriate Adomian's polynomials. These \( A_n \) polynomials depend on the particular nonlinearity and these \( A_n \) Adomian polynomials are calculated by the general formula

\[
A_n(u_0, u_1, ..., u_n) = \frac{1}{n!} \left[ \frac{d^n}{d\lambda^n} \left( N \left( \sum_{k=0}^{n} \lambda^k u_k \right) \right) \right]_{\lambda=0}, \quad n \geq 0
\]

(10)

Substituting eq. (8) and eq. (9) into eq. (5) gives

\[
\sum_{n=0}^{\infty} u_n = \Phi + L^{-1} g - L^{-1} R \sum_{n=0}^{\infty} u_n - L^{-1} \sum_{n=0}^{\infty} A_n
\]

(11)

Each term of series (8) is given by the recurrence relation

\[
\begin{align*}
    u_0 &= \Phi + L^{-1} g \\
    u_1 &= -L^{-1} Ru_0 - L^{-1} A_0 \\
    u_2 &= -L^{-1} Ru_1 - L^{-1} A_1 \\
    u_3 &= -L^{-1} Ru_2 - L^{-1} A_2 \\
    &\vdots \\
    u_n &= -L^{-1} Ru_{n-1} - L^{-1} A_{n-1}
\end{align*}
\]

(12)

where \( A_n \) are the special Adomian polynomials or equivalently

\[
\begin{align*}
    A_0 &= f(u_0), \\
    A_1 &= u_1 f'(u_0), \\
    A_2 &= u_2 f'(u_0) + \frac{1}{2} u_2^2 f''(u_0) \\
    A_3 &= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{1}{3!} u_3^3 f'''(u_0) \\
    A_4 &= u_4 f'(u_0) + \left( u_1 u_3 + \frac{1}{2} u_2^2 \right) f''(u_0) + \frac{1}{2} u_1^2 u_2 f'''(u_0) + \frac{1}{3!} u_4^4 f^{(4)}(u_0) + \frac{1}{24} u_4^4 f^{(4)}(u_0)
\end{align*}
\]

(13)

So, the practical solution for the \( n \)-term approximation is,

\[
\Phi_n = \sum_{i=0}^{n-1} u_i, \quad n \geq 1
\]

(14)

and the exact solution is
\[ u = \lim_{n \to \infty} \varphi_n = \sum_{i=0}^{\infty} u_i \quad (15) \]

3. The Adomian decomposition method applied to Newell-Whitehead model

The nonlinear wave equation with dissipation and nonlinear transport term is given as: [15], [16]

\[ \alpha u_{xx} + \beta u_x + \gamma u_t = u_{xx} - f(u) \quad ; \quad f(u) = (u - a_1)(u - a_2)(u - a_3) \quad (16) \]

Where \( a_1, a_2, a_3 \) are distinct real numbers and \( \alpha, \beta, \gamma \), are constants.

For \( \alpha = 0, \beta = 0, \gamma = 1 \), Eq. (16) reduces to the nonlinear reaction-diffusion form

\[ u_t = u_{xx} - f(u) \quad ; \quad f(u) = (u - a_1)(u - a_2)(u - a_3) \quad (17) \]

for different choices of the parameters \( a_1, a_2, a_3 \) Eq. (17) reduces to the well known nonlinear reaction diffusion equations appearing in many different branches of sciences: when \( a_3 = 0, a_2 = 1, a_1 = -1 \) we get the Newell-Whitehead equation [20]

\[ \frac{\partial u}{\partial t} = \Delta u + u(1 - u^2) \quad (t, x) \in (0, \infty) \times \Omega \quad (18) \]

with the initial and boundary conditions

\[ u(x,0) = u_0(x), \quad x \in \Omega \]
\[ u(x,t) = 0, \quad t > 0, \quad x = 0 \quad \text{and} \quad x = L \quad (18a) \]
\[ \frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = L. \]

which arises after carrying out a suitable normalization in the study of thermal convection of a fluid heated from below. Considering the Perturbation from a stationary state, the equation describes the evolution of the amplitude of the vertical velocity if this is a slowly varying function of time \( t \) and position \( x. \)

The equation (18) written in an operator form

\[ L_u = u_{xx} + u - N(u) \quad (19) \]

where \( L = \frac{\partial}{\partial t} \) and \( N(u) = u^3 \) is nonlinear term. Applying the inverse operator to the equation (19) and using the initial data (18a) yields

\[ u(x,t) = u_0(x) + L^{-1}u_{xx} + L^{-1}u - L^{-1}N(u) \quad (20) \]

The ADM suggests the solution \( u(x,t) \) be decomposed by infinite series of components

\[ u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (21) \]

and the nonlinear operator \( N(u) \) by the infinite series of the Adomian polynomials

\[ N(u) = \sum_{n=0}^{\infty} A_n \quad (22) \]

The first four components of Adomian polynomials according to (13) read

\[ A_0 = u_0^3 \]
An Approximate Solution to The Newell-Whitehead Equation by …

\[ A_1 = 3u_0^3 u_1 \]
\[ A_2 = 3u_0^3 u_2 + 3u_0 u_1^2 \]
\[ A_3 = 3u_0^3 u_3 + 6u_0 u_1 u_2 + u_1^3 \] (23)

From eq. (21) and eq. (22), the iterates are determined by the following recursive way:

\[ u_0 = u_0(x) \]
\[ u_n = L^{-1}(\Delta u_{n-1} + u_{n-1}) - L^{-1}(A_{n-1}) \quad n \geq 1. \] (24)

The decomposition method provides a reliable technique that requires less work if compared with traditional techniques.

4. Application and Numerical Results

To give a clear overview of the methodology, the following example will be discussed. All the results are calculated by using the MATLAB 7.4 software.

Consider the Newell-Whitehead equation [11]
\[ \frac{\partial u}{\partial t} = \Delta u + u(1-u^2) \]

with the initial conditions
\[ u(x,0) = \frac{a}{b} \frac{c_1 e^{(0.5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + 1)} \] (25)

And exact Solution
\[ u(x,t) = \frac{a}{b} \frac{c_1 e^{(0.5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + 1)} \]

Substitution the initial condition eq. (25) into eq. (24) and using eq. (23) to calculate the Adomian polynomials, yields the following recursive relation

\[ u_n = L^{-1}(\Delta u_{n-1} + u_{n-1}) - L^{-1}(A_{n-1}) \quad n \geq 1 \] (26)

We will take \( a = b = c_1 = c_2 = c_3 = 1 \). The first few terms of the decomposition series are given by:
\[ u_0 = \frac{a}{b} \frac{c_1 e^{(0.5(\sqrt{2a})x)} - c_2 e^{(-0.5(\sqrt{2a})x)}}{(c_1 e^{(0.5(\sqrt{2a})x)} + c_2 e^{(-0.5(\sqrt{2a})x)} + 1)} \]
\[ u_1 = \frac{3/2}{a} \left( \frac{e^{(1/2\sqrt{3})x}}{e^{(1/2\sqrt{3})x} + e^{(-1/2\sqrt{3})x}} + 1 \right) \]
\[ u_2 = \frac{-3/2}{a} \left( \frac{e^{(1/2\sqrt{3})x}}{e^{(1/2\sqrt{3})x} + e^{(-1/2\sqrt{3})x}} + 1 \right)^2 \]
\[ u_3 = \frac{9/16}{a} \left( \frac{e^{(1/2\sqrt{3})x}}{e^{(1/2\sqrt{3})x} + e^{(-1/2\sqrt{3})x}} + 1 \right)^3 \] (27)
other components are determined similarly (we will use four).
Substituting relations (27) into recursive relation (21) yields

\[
\begin{align*}
    u &= -e^{(\frac{1}{2}\tau x)} + 3/2((e^{(\frac{3}{2}\tau x)} + e^{(\frac{1}{2}\tau x)})t)/((e^{(\frac{1}{2}\tau x)} + e^{(\frac{1}{2}\tau x)} + 1)^2) \\
    &+ \frac{9/16(t^2((e^{(\frac{3}{2}\tau x)} + e^{(\frac{1}{2}\tau x)})(e^{(\frac{1}{2}\tau x)} - \beta + \frac{3}{2}e^{(\frac{1}{2}\tau x)} - e^\tau e^{(\frac{3}{2}\tau x)} + \frac{3}{2}e^{(\frac{1}{2}\tau x)}))}{(e^{(\frac{1}{2}\tau x)} + e^{(\frac{1}{2}\tau x)} + 1)^2)}
\end{align*}
\]

(28)

Fig.1: Comparison of exact and Adomain solution for $t=0.5$

Fig.2: Comparison of exact and Adomain solution for $t=1.0$
An Approximate Solution to The Newell-Whitehead Equation by …

Table 1: Comparison of the exact and Adomain decomposition solutions

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<th>Adomain Decomposition $u(x,t)$</th>
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</tbody>
</table>

It is clear from the figures (1-3) and the table that Adomain decomposition method is so accurate and converging to the exact solution.
5. Conclusions

The Adomain decomposition method is effective and powerful method for solving nonlinear partial differential Newell-Whitehead equations. The important part of this method is calculating Adomain polynomials for nonlinear operator.

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REFERENCES


