On Rings whose Simple Singular R-Modules are
GP-Injective

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ABSTRACT

In this work we give a characterization of rings whose simple singular right R-modules are Gp-injective. We prove that if R is a quasi-duo ring whose simple singular right R-modules are Gp-injective, then any reduced right ideal of R is a direct summand. We also consider that a zero commutative ring with every simple singular left R-module is Gp-injective.
1. Introduction:
Throughout this paper, R denotes an associative ring with identity, and all modules are unitary right R-modules. Recall that:
(1) A right R-module M is called general right principally injective (briefly right Gp-injective) if for any \(0 \neq a \in R\) there exists a positive integer \(n\), such that \(a^n \neq 0\) and any right R-homomorphism of \(a^n R\) into M extends to one of R into M;
(2) R is called reduced if R has no non-zero nilpotent elements;
(3) R is right (left) quasi-duo ring if every maximal right (left) ideal of R is an ideal of R;
(4) A ring R is called semi-prime if 0 is the only nilpotent ideal;
(5) for any element a in R we define a right annihilator of a by \(r(a) = \{x \in R : ax = 0\}\) and a left annihilator of a, \(l(a)\) is similarly defined.

2. Rings whose simple singular modules are GP-Injective:
In this section, we study rings whose simple singular right R-modules are Gp-injective.
We begin this section with the following result.

**Proposition 2-1:**
Let R be a quasi-duo ring, with every simple singular right R-modules is Gp-injective. Then any reduced right ideal of R is a direct summand.

**Proof:** Let \(I = aR\) be a reduced principal right ideal of R. We shall show that \(aR + r(a) = R\). If not, there exists a maximal right ideal \(M\) of R such that \(aR + r(a) \subseteq M\). Now, \(M\) is essential right ideal of R, if not, then there exists a non-zero right ideal \(L\) of R such that \(ML = 0\). Then \(aRL \subseteq ML \subseteq M\), implies that \(L \subseteq r(a) \subseteq M\), so \(ML = L = 0\), and this is a contradiction.

So \(M\) must be essential right ideal of R. Therefore \(R/M\) is Gp-injective. Then there exists a positive integer \(n\) such that any R-homomorphism of \(a^n R\) into \(R/M\) extends to one of R into \(R/M\). Let \(f : a^n R \rightarrow R/M\) be defined by \(f(a^n r) = r + M\). \(f\) is a well-defined R-homomorphism. Indeed, let \(r_1, r_2 \in R\) such that \(a^n r_1 = a^n r_2\). Then \(a^n (r_1 - r_2) = 0\), implies that \(a^n (r_1 - r_2) = 0\), so \(r_1 - r_2 \in r(a^n)\), since I is reduced. Therefore \(r(a^n) = r(a)\), this implies that \(r_1 - r_2 \in r(a) \subseteq M\). Hence, \(r_1 + M = r_2 + M\). Now \(R/M\) is Gp-injective, so there exists \(c \in R\) such that \(1 + M = f(a^n) = ca^n + M\). Hence, \(1 - ca^n \in M\), since \(a^n \in M\) and R is a quasi-duo ring, then \(ca^n \in M\) and so \(1 \in M\). This contradicts \(M \neq R\).

Therefore \(aR + r(a) = R\). In particular \(ar + c = 1\), for some \(r \in R\) and \(c \in r(a)\), whence \(a^2 r = a\). If we set \(d = ar^2 \in I\), then \(a = a^2 d\). Clearly \((a - ada)^2 = 0\), since I is reduced, thus \(a = ada\), and hence \(I = eR\), where \(e = ad\) is an idempotent element. Thus I is a direct summand.
**Proposition 2-2:**

Let $R$ be a semi-prime ring with every simple singular right $R$-module is Gp-injective. Then every right ideal of $R$ is an idempotent.

**Proof:** For any right ideal $I$ of $R$, suppose there exists an element $b$ in $I$, such that $b \notin I^2$. Then $bR \neq (bR)^2$. Since $R$ is a semi-prime ring, then $(bR)^2$ is essential in $bR$. By Zorn’s lemma, the set of right ideals $J$ such that $(bR)^2 \subseteq J \subseteq bR$ has a maximal member $L$. Then $bR/L$ is a simple singular, and therefore is Gp-injective. Now, let $f: bR \rightarrow bR/L$ is the canonical homomorphism defined by $f(br) = br + L$ for all ring $R$, since $bR/L$ is Gp-injective, so there exists $c \in R$, such that $f(br) = (bc + L)br$. Then $f(b) = (bc + L)b = b + L$, which implies that $b + L = bc + L$. Hence, $b - bc \notin L$, whence it follows that $b \in L$. Thus $bR \subseteq L$ and this is a contradiction. Therefore $I = I^2$.

3-Zero Commutative Rings

In this section we introduce the notion of a zero commutative ring in order to study the connection between rings whose simple singular right $R$-modules are Gp-injective and other rings.

**Definition 3-1:**

A ring $R$ is called zero commutative (briefly ZC) if for $a, b \in R$, $ab = 0$ if $ba = 0$.

We shall begin this section with the following result.

**Lemma 3-2:**

Let $R$ be a ZC ring. Then $RaR + l(a)$ is an essential left ideal of $R$.

**Proof:** Given $a \in R$, assume that $[RaR + l(a)]l = 0$, where $I$ is a right ideal of $R$. Then $al \subseteq I$ $RaR = 0$, so $l \subseteq r(a) \subseteq l(a)$. Hence, $I = 0$; where $RaR + l(a)$ is an essential left ideal of $R$.

**Lemma 3-3:**

Let $R$ be a ZC ring with every simple singular left $R$-module is Gp-injective, then $R$ is reduced.

**Proof:** Let $a^2 = 0$. Suppose that $a \neq 0$. By lemma (3-2), $I(a)$ is an essential left ideal of $R$, since $a \neq 0$, $l(a) \neq R$. Thus, there exists a maximal essential left ideal $M$ of $R$ containing $l(a)$, therefore $R/M$ is Gp-injective. So any $R$-homomorphism of $Ra$ into $R/M$ extends to one of $R$ into $R/M$. Let $f: Ra \rightarrow R/M$ be defined by $f(ra) = r + M$. Clearly, $f$ is a well-defined $R$-
homomorphism. Thus \( 1+M = f(a) = ac+M \). Hence, \( 1-ac \in M \) and so \( 1 \in M \), which is a contradiction. Hence \( a=0 \), and so \( R \) is reduced.

**Definition 3-4:**
A ring \( R \) is said to be right weakly regular if for all \( a \in R \), there exists \( b \in RaR \) such that \( a=ab \).

Now, we give the main result.

**Proposition 3-5:**
If \( R \) is ZC and every simple singular left \( R \)-module is Gp-injective, then \( R \) is a reduced weakly regular ring.

**Proof:** By Lemma (3-3), \( R \) is a reduced ring. We shall show that \( RaR+l(a)=R \) for any \( a \in R \). Suppose that there exists \( b \in R \) such that \( RbR+l(b) \neq R \). Then there exists a maximal left ideal \( M \) of \( R \) containing \( RbR+l(b) \). By Lemma (3-2), \( M \) must be essential in \( R \). Therefore \( R/M \) is Gp-injective. So there exists a positive integer \( n \) such that any \( R \)-homomorphism of \( Rb^n \) into \( R/M \) extends to one of \( R \) into \( R/M \). Let \( f:Rb^n \rightarrow R/M \) be defined by \( f(rb^n)=r+M \). Since \( R \) is a reduced ring, \( f \) is a well-\( R \)-homomorphism. Now, \( R/M \) is Gp-injective, so there exists \( c \in R \) such that \( 1+M = f(b^n) = b^n c + M \). Hence \( 1-b^n c \in M \) and so \( 1 \in M \), which is a contradiction. Therefore \( RaR+l(a)=R \) for any \( a \in R \). Hence \( R \) is a left weakly regular ring. Since \( R \) is reduced, then \( RaR+r(a)=R \), implies that \( R \) is a right weakly regular ring. Therefore \( R \) is a weakly regular ring.

**Kim and Nam in [2] proved that.** Rings whose simple right \( R \)-modules are Gp-injective are always semi-prime. But in general rings whose simple singular right \( R \)-modules are Gp-injective need not be semi-prime.

**Proposition 3-6:**
Let \( R \) be a ZC ring, and every simple singular left \( R \)-module is Gp-injective, then \( R \) is a semi-prime ring.

**Proof:** From Lemma (3-3), \( R \) is a reduced ring and then \( R \) is a semi-prime ring.


REFERENCES


