**The n-Wiener Polynomials of the Cartesian Product of a Complete Graph with some Special Graphs**

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**ABSTRACT**

The n-Wiener polynomials of the Cartesian products of a complete graph $K_t$ with another complete graph $K_r$, a star graph $S_r$, a complete bipartite graph $K_{t,s}$, a wheel $W_r$, and a path graph $P_r$ are obtained in this paper. The n-diameters and the n-Wiener indices of $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{r,s}$, $K_t \times W_r$, and $K_t \times P_r$ are also obtained.

Keywords: n-distance, n-diameter, n-index, n-Wiener polynomial.

**1. Introduction.**

We follow the terminology of [5] and [6]. Let $v$ be a vertex of a connected graph $G$ and let $S$ be an $(n-1)$-subset of vertices of $V(G)$, $n \geq 2$, then the **n-distance** $d_n(v,S)$ is defined as follows[7]

$$d_n(v,S) = \min \{d(v,u) : u \in S\}. \quad ...(1.1)$$

Sometimes, we refer to the n-distance of the pair $(v,S)$ in $G$ by $d_n(v,S | G)$. The **n-diameter** $\text{diam}_n G$ of $G$ is defined by

$$\text{diam}_n G = \max \{d_n(v,S) : v \in V(G), S \subseteq V(G), |S| = n-1\}. \quad ...(1.2)$$

It is clear that for all $2 \leq m \leq n \leq p$,

$$\text{diam}_n G \leq \text{diam}_m G \leq \text{diam}_p G. \quad ...(1.3)$$

The **n-Wiener index** of $G$ denoted by $W_n(G)$ is defined as

$$W_n(G) = \sum_{(v,S)} d_n(v,S), \quad ... (1.4)$$

where the summation is taken over all pairs $(v,S)$ for which $v \in V(G)$, $S \subseteq V(G)$ and $|S| = n-1$. The **n-average distance** $\mu_n(G)$ is defined as
\[
\mu_n(G) = \frac{W_n(G)}{p^{n-1}}, \quad 3 \leq n \leq p. \quad \ldots (1.5)
\]

Let \( v \) be any vertex of \( G \), then the **n-distance of** \( v \) denoted \( d_n(v|G) \) or simply \( d_n(v) \) is defined as
\[
d_n(v) = \sum_{S \subseteq V(G), |S| = n-1} d_n(v,S). \quad \ldots (1.6)
\]

The Wiener polynomial of \( G \) with respect to the n-distance, which is called n-Wiener polynomial and defined as below.

**Definition 1.1.**[2]. Let \( C_n(G,k) \) be the number of pairs \( (v,S) \), \( |S|=n-1,3 \leq n \leq p \), such that \( d_n(v,S)=k \), for each \( 0 \leq k \leq \delta_n \). Then, **the n-Wiener polynomial** \( W_n(G;x) \) is defined by
\[
W_n(G;x) = \sum_{k=0}^{\delta_n} C_n(G,k)x^k, \quad \ldots (1.7)
\]
in which \( \delta_n \) is the n-diameter of \( G \).

One may easily see [2] that for \( 3 \leq n \leq p \), the number of all \( (v,S) \) pairs is
\[
p^{\binom{n-1}{1}}, \text{ and } [1]
\]
\[
\sum_{k=0}^{\delta_n} C_n(G,k) = \binom{p-1}{n-1}, \quad C_n(G,0) = \binom{p-1}{n-2}, \quad \ldots (1.8)
\]
\[
C_n(G,1) = p^{\binom{n-1}{1}} - \sum_{v \in V(G)} \binom{p-1-\deg_{G}(v)}{n-1}, \quad \ldots (1.9)
\]

**Definition 1.2**[1] Let \( v \) be a vertex of \( G \), and let \( C_n(v,G,k) \) be the number of \( (n-1) \)-subsets of vertices of \( G \) such that
\[
d_n(v|S|G) = k, \quad \text{for } n \geq 3, \quad 0 \leq k \leq \delta_n.
\]

Then, the **n-Wiener polynomial of vertex** \( v \), denoted by \( W_n(v,G;x) \) is defined as
\[
W_n(v,G;x) = \sum_{k \geq 0} C_n(v,G,k)x^k. \quad \ldots (1.10)
\]

It is clear that for all \( k \geq 0 \),
\[
\sum_{v \in V(G)} C_n(v,G,k) = C_n(G,k), \quad \ldots (1.11)
\]
and
\[
\sum_{v \in V(G)} W_n(v,G;x) = W_n(G;x). \quad \ldots (1.12)
\]
There are many classes of graphs $G$ in which for each $k, 1 \leq k \leq \delta_n$, $C_n(v, G, k)$ is the same for every vertex $v \in V(G)$; such graphs are called [1] **vertex-n-distance regular**. If $G$ is of order $p$ and it is vertex-n-distance regular, then
\[
W_n(G; x) = pW_n(v, G; x),
\]
where $v$ is any vertex of $G$.

The authors of references [2], [3] and [4] obtained the $n$-Wiener polynomials of some special graphs and some types of composite graphs. In this paper, we obtain $n$-Wiener polynomials of the Cartesian products $K_t \times K_r$, $K_t \times S_r$, $K_t \times K_{r,s}$, $K_t \times W_r$ and $K_t \times P_r$.

### 2. The Cartesian Product of a Complete Graph and a Star

Let $K_t$ be a complete graph with $V(K_t) = \{u_1, u_2, \ldots, u_t\}$, and $S_r$ be a star of center $v_0$ and end vertices $v_1, v_2, \ldots, v_{r-1}$. Each vertex of $K_t \times S_r$ is an ordered pair $(u_i, v_j)$, $1 \leq i \leq t$, $0 \leq j \leq r - 1$. Let $K^i_t$ be the clique graph [6] of order $t$ of vertex set $\{(u_i, v_j): i = 1, 2, \ldots, t, 0 \leq j \leq r - 1\}$. The graph $K_t \times S_r$ is depicted in Fig. 2.1.

![Fig. 2.1. The graph $K_t \times S_r$.](image)

It is clear that $0 \leq d((u_i, v_j), (u_i, v_m)) \leq 3$. Thus,
\[
\text{diam}_n K_t \times S_r \leq \text{diam} K_t \times S_r \leq 3.
\]

**Proposition 2.1.** For $t \geq 2$, $r \geq 3$, the $n$-diameter of $K_t \times S_r$ is given by
\[
\text{diam}_n K_t \times S_r = \begin{cases} 
3, & \text{if } 2 \leq n \leq (t-1)(r-2)+1, \\
2, & \text{if } 1+(t-1)(r-2)<n\leq t(r-1), \\
1, & \text{if } t(r-1)<n\leq rt.
\end{cases}
\]

**Proof.** The proof is clear from Fig. 2.1. ■
The following theorem gives us the n-Wiener polynomial of \( K_t \times S_r \). It is clear that the order of \( K_t \times S_r \) is \( p=rt \).

**Theorem 2.2.** For \( t \geq 2, r \geq 3, 3 \leq n \leq rt \),

\[
W_n(K_t \times S_r; x) = p \sum_{n=2}^{p-1} \left[ \binom{p-1}{n-1} x^{n-1} \right] + t \left[ \binom{p-t-r+1}{n-1} x^{n-1} \right] + (r-1) \left[ \binom{p-t-1}{n-1} x^{n-1} \right]
\]

\[
+ (r-1) \left[ \binom{p-2(r-2)}{n-1} x^{n-1} \right].
\]

**Proof.** It is clear that each vertex of \( K_t^0 \) is of degree \( t+r-2 \), and each vertex of \( K_t^j, 1 \leq j \leq r-1 \), is of degree \( t \). Therefore, by (1.9) we obtained \( C_n(K_t \times S_r,1) \) as given in the theorem.

To find \( C_n(K_t \times S_r,3) \), we notice that there are \( (t-1)(r-2) \) vertices each of distance 3 from each vertex \((u,v)\) of \( K_t^j, 1 \leq j \leq r-1 \). Thus,

\[
C_n(K_t \times S_r,3) = t(r-1) \binom{p-t-1}{n-1} x^{n-1} + (r-1) \binom{p-2(r-2)}{n-1} x^{n-1}.
\]

Finally, by (1.8) and Proposition 2.1, we get

\[
C_n(K_t \times S_r,2) = p \binom{p-1}{n-1} - C_n(K_t \times S_r,1) - C_n(K_t \times S_r,3)
\]

\[
= t \left[ \binom{p-t-r+1}{n-1} x^{n-1} \right] + (r-1) \left[ \binom{p-t-1}{n-1} x^{n-1} \right] + (r-1) \left[ \binom{p-2(r-2)}{n-1} x^{n-1} \right].
\]

Hence, the proof.\( \blacksquare \)

**Corollary 2.3.** For \( t \geq 2, r \geq 3, 3 \leq n \leq rt \),

\[
W_n(K_t \times S_r) = p \sum_{n=2}^{p-1} \left[ \binom{p-1}{n-1} x^{n-1} \right] + t \left[ \binom{p-t-r+1}{n-1} x^{n-1} \right] + (r-1) \left[ \binom{p-t-1}{n-1} x^{n-1} \right] + (r-1) \left[ \binom{p-2(r-2)}{n-1} x^{n-1} \right].
\]

**3. The Cartesian Product of Complete Graphs**

Let \( K_t \) and \( K_r \) be disjoint complete graphs, and let \((u_1,v_1),(u_2,v_2) \in V(K_t \times K_r)\), then it is clear that \( \text{diam} K_t \times K_r = 2 \).

Thus,

\[
\text{diam}_n K_t \times K_r \leq 2, \quad 2 \leq n \leq rt.
\]

If \( u_1 \neq u_2 \) and \( v_1 \neq v_2 \), then \((u_1,v_1), (u_2,v_2)\) are non-adjacent in \( K_t \times K_r \); and \((u_1, v_1), (u_1,v_2), (u_2,v_2)\) is a path of length 2. Therefore,
The degree of each vertex \((u_1,v_1)\) is \(r+t-2\). Thus, the number of vertices of distance 2 from \((u_1,v_1)\) is \(rt-r-t+1\). Hence, we have the following result.

**Proposition 3.1.** For \(t,r \geq 2\),

\[
\text{diam}_n K_t \times K_r = \begin{cases} 
2 & \text{if } \ 2 \leq n \leq rt-r-t+2, \\
1 & \text{if } \ rt-r-t+3 \leq n \leq rt.
\end{cases}
\]

Now, we find the \(n\)-Wiener polynomial of \(K_t \times K_r\).

**Theorem 3.2.** For \(r,t \geq 2, 3 \leq n \leq rt\)

\[
W_n(K_t \times K_r; x) = rt \binom{n-1}{n-2} + \frac{2^{n-r-t+1}}{n-1} \binom{n-1}{n-2} x + rt \frac{2^{n-r-t+1}}{n-1} x^2.
\]

**Proof.** It is clear that \(K_t \times K_r\) is vertex-\(n\)-distance regular. Thus,

\[
C_n(K_t \times K_r, 2) = rt C_n((u_1,v_1), K_t \times K_r, 2).
\]

Since the number of vertices of distance 2 from \((u_1,v_1)\) is \(rt-r-t+1\), and there is no vertex of distance more than 2 from \((u_1,v_1)\), then

\[
C_n((u_1,v_1), K_t \times K_r, 2) = \binom{n-r-t+1}{n-1}.
\]

The constant term and the coefficient of \(x\) follow from (1.8) and (1.9). 

**Corollary 3.3.** For \(r,t \geq 2, 3 \leq n \leq rt\),

\[
W_n(K_t \times K_r) = rt \binom{n-1}{n-2} + \frac{2^{n-r-t+1}}{n-1} \binom{n-1}{n-2}.
\]

4. The Cartesian Product of a Complete Graph and a Complete Bipartite Graphs

Let \(K_{r,s}\) be a complete bipartite graph of bipartite sets of vertices \(V_1 = \{v_1,v_2,\ldots,v_r\}, V_2 = \{w_1,w_2,\ldots,w_s\}; r \geq s\), and let \(V(K_t) = \{u_1,u_2,\ldots,u_t\}\), then it is clear that in \(K_t \times K_{r,s}\)

\[
d((u_i,v_h),(u_j,v_k)) = 3 \text{ when } i \neq j, h \neq k,
\]

because there is a shortest path

\((u_i,v_h), (u_j,v_h), (u_j,w), (u_j,v_k), w \in V_2\).

Similarly,

\[
d((u_i,w_h),(u_j,w_k)) = 3 \text{ when } i \neq j, h \neq k.
\]

Moreover,

\[
d((u_i,v_h),(u_i,v_k)) = d((u_i,w_h),(u_i,w_k)) = 2.
\]

Therefore,

\[
\text{diam } K_t \times K_{r,s} = 3.
\]
and so
\[ \text{diam}_n K_t \times K_{r,s} \leq 3, \ 2 \leq n \leq p, \ p=tr(s) \].

For any vertex \((u_i, v_h)\), the number of vertices of distance 3 from \((u_i, v_h)\) in \(K_t \times K_{r,s}\) is \((t-1)(r-1)\). Similarly, there are \((t-1)(s-1)\) vertices of distance 3 from \((u_i, w_k)\). Moreover, the degree of each vertex of \(K_t \times K_{r,s}\) is either \(r+t-1\) or \(s+t-1\).

Thus, we have the following result.

**Proposition 4.1.** For \(t, r, s \geq 2, r \geq s\), then the n-diameter of \(K_t \times K_{r,s}\) is given
\[
\text{diam}_n K_t \times K_{r,s} =
\begin{cases} 
3, & \text{for } 2 \leq n \leq tr-t-r+2, \\
2, & \text{for } tr-t-r+3 \leq n \leq p-t-s, \\
1, & \text{for } p-t-s+1 \leq n \leq p.
\end{cases}
\]

The next theorem determines the n-Wiener polynomial of \(K_t \times K_{r,s}\).

**Theorem 4.2.** For \(t, r, s \geq 2, 3 \leq n \leq p, p=tr(s)\),
\[
W_n(K_t \times K_{r,s}; x) = \left( \begin{array}{c} p-1 \\ n-2 \end{array} \right) + [p \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) - rt \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) - \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right)] x + [rt \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) - \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right)] x^2 + [rt \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + st \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right)] x^3.
\]

**Proof.** \(C_n(K_t \times K_{r,s}, 0)\) and \(C_n(K_t \times K_{r,s}, 1)\) are obtained from (1.8) and (1.9). To find the other coefficients, we notice that \(C_n((a,b), K_t \times K_{r,s}, k)\) is the same for every vertex \((a,b)\)\(\in V(K_t)\times V_1\), and \(C_n((c,d), K_t \times K_{r,s}, k)\) is the same for every vertex \((c,d)\)\(\in V(K_t)\times V_2\), for \(k=2,3\). Since the number of vertices of distance 3 from vertex \((a,b)\) is \((t-1)(r-1)\), and the number of vertices of distance 3 from vertex \((c,d)\) is \((t-1)(s-1)\), then we get the coefficient of \(x^3\) as given in the statement of the theorem.

Finally, \(C_n(K_t \times K_{r,s}, 2)\) is obtained using the relation (1.8) and the coefficients already obtained. This completes the proof. ■

**Corollary 4.3.** For \(t, r, s \geq 2, 3 \leq n \leq p\) in which \(p=tr(s)\),
\[
W_n(K_t \times K_{r,s}; x) = \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + rt \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + st \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right) + \left( \begin{array}{c} p-1 \\ n-1 \end{array} \right).
\]
Proof. The n-Wiener index is obtained from $W_n(K_t \times K_{r,s}; x)$ by taking the derivative with respect to $x$, and then put $x=1$, and simplified the expression.

5. The Cartesian Product of a Complete Graph and a Wheel

Let $W_r$ be a wheel of order $r \geq 4$ and let its center be denoted by $v_0$ and its other vertices be $v_1, v_2, \ldots, v_{r-1}$. Moreover, let $V(K_t) = \{u_1, u_2, \ldots, u_t\}$. The order of $K_t \times W_r$ is $p=rt$, and in $K_t \times W_r$

$$\text{deg}(u_i, v_j) = t+2, \text{ for } 1 \leq i \leq t, 1 \leq j \leq r-1,$$

$$\text{deg}(u_i, v_0) = t+r-2.$$

One can easily see that in $K_t \times W_r$

$$d((u_i, v_0), (u_j, v_h)) = 2, \text{ for } i \neq j, h \neq 0,$$

$$d((u_i, v_h), (u_j, v_m)) = 3, \text{ for } i \neq j, h \neq m, h, m \neq 0,$$

because $(u_i, v_h), (u_j, v_h), (u_j, v_0), (u_j, v_m)$ is a shortest $(u_i, v_h) - (u_j, v_m)$ when $v_h v_m \notin W_r$. Thus, $diam K_t \times W_r = 3$, when $r \geq 5$.

Thus, for $r \geq 5$, $t \geq 2$,

$$diam_n K_t \times W_r \leq 3.$$

Since for each vertex $(u_i, v_h)$, $1 \leq i \leq t$, $h \neq 0$ there are $(t-1)(r-4)$ vertices of distance 3 from $(u_i, v_h)$, and $\text{deg}(u_i, v_h) = t+2$, then we have the following result.

Proposition 5.1. For $t \geq 2$, $r \geq 5$, the n-diameter of $K_t \times W_r$ is given by

$$diam_n K_t \times W_r = \begin{cases} 3, & \text{for } 2 \leq n \leq 1+(t-1)(r-4), \\ 2, & \text{for } 2+(t-1)(r-4) \leq n \leq p-t-2, \\ 1, & \text{for } p-t-1 \leq n \leq p. \end{cases}$$

The following theorem gives us the n-Wiener polynomial of $K_t \times W_r$.

Theorem 5.2. For $t \geq 2$, $r \geq 5$, $3 \leq n \leq p$, $p=tr$

$$W_n(K_t \times W_r; x) = p \left[ \sum_{n=2}^{p-1} \binom{n}{n-1}^{t(r-1)-t} \binom{n}{n-1}^{t(r-1)+t} x^{n} + \binom{n}{n-1}^{t(r-1)-t} \binom{n}{n-1}^{t(r-1)+t} x^{n^2} \right] x^{3}. $$

Proof. The coefficients of $x^0$ and $x$ are obtained using (1.8) and (1.9). To obtain the coefficient of $x^3$, we notice that for any $(u_i, v_0)$, $1 \leq i \leq t$ and every $(n-1)$-set of vertices $S$, $d_a((u_i, v_0), S) = 2$. But for every vertex $(u_i, v_j)$, $1 \leq i \leq t$, $1 \leq j \leq r-1$, there are $(t-1)(r-4)$ vertices each of distance 3 from $(u_i, v_j)$. 123
Therefore, there are \( \binom{p-r-4t+4}{n-1} \) sets \( S, |S|=n-1 \), such that \( d_n((u_i,v_j),S)=3 \).

Thus,

\[
C_n(K_t \times W_r,3) = t(r-1) \binom{p-r-4t+4}{n-1}.
\]

We obtain \( C_n(K_t \times W_r,2) \) by using (1.8). Hence, the proof. ■

**Corollary 3.4.3.** For \( t \geq 2, r \geq 5 \) and \( 3 \leq n \leq rt \),

\[
W_n(K_t \times W_r) = t(r-1) \binom{p-r-4t+4}{n-1} + t \binom{p-r-t+1}{n-1} + t(r-1) \binom{p-r-4t+4}{n-1}.
\]

**Proof.** The proof follows from Theorem 5.2 and the fact \( W_n(K_t \times W_r) = W_n(K_t \times W_r;1) \). ■

6. The Cartesian Product of a Path and a Complete Graph

Let \( P_r, r \geq 2 \) be a path graph of order \( r \) and \( P_r: v_1,v_2,\ldots,v_r \), and let \( V(K_t) = \{u_1,u_2,\ldots,u_t\}, t \geq 3 \).

The Cartesian product \( K_t \times P_r \) is shown in Fig. 6.1. The following proposition determines the n-diameter of \( K_t \times P_r \).

**Proposition 6.1.** For \( r \geq 2, t \geq 3, 2 \leq n \leq rt \),

\[
diam_n K_t \times P_r = r+1 - \left\lceil \frac{n}{t} \right\rceil.
\]

![Fig. 6.1. The graph \( K_t \times P_r \)](image-url)
Proof. From Fig. 6.1, we notice that $d_n(u,v,S)$, $|S|=n-1$ has maximum value when $(u,v)$ is one of the vertices in $A_1 \cup A_t$, where

$$A_i = \{(u_j,v_i) : j=1,2,\ldots,t\},$$

and $S$ is the $(n-1)$-set of vertices farthest from $(u,v)$ in $K_t \times P_r$. Thus, we may take the vertex $(u_1,v_r)$, and $S$ consisting of vertices of $A_1,A_2,\ldots,A_i$ and some vertices of $A_{i+1}-\{(u_1,v_{i+1})\}$ when

$$i \leq n-1 \leq (i+1)t-1;$$

and when

$$2 \leq n \leq t,$$

then $S \subseteq A_1-\{(u_1,v_1)\}$.

In the last case,

$$\text{diam}_n K_t \times P_r = r;$$

and in general case of $n$,

$$\text{diam}_n K_t \times P_r = r-i, \; \; i+1 \leq n \leq (i+1)t.$$  

One can easily see that

$$i = \lceil \frac{n}{t} \rceil - 1.$$  

Hence, in any case of the value of $n$,

$$\text{diam}_n K_t \times P_r = r+1 - \lfloor \frac{n}{t} \rfloor.$$  

Now, we obtain the $n$-Wiener polynomial of $K_t \times P_r$ in the following two theorems.

**Theorem 6.2.** Let $r=2s$, $s \geq 1$, $t \geq 3$ and $3 \leq n \leq rt$. Then

$$W_n(K_t \times P_r;x) = \sum_{k=0}^{\delta} C_n(K_t \times P_r,k)x^k,$$

where

$$C_n(K_t \times P_r,0) = rt \binom{n-1}{n-2},$$

$$C_n(K_t \times P_r,1) = rt \binom{n-1}{n-1} - 2t \binom{n-1}{n-1} - t(r-2) \binom{n-1}{n-1},$$

for $2 \leq k \leq s$

$$C_n(K_t \times P_r,k) = 2t \sum_{i=1}^{k-1} \binom{a+i-r-1}{n-1} - \binom{a-r}{n-1} + 2 \binom{a+2t-r-1}{n-1} - \binom{a}{n-1},$$

$$+ (s-k) \left( \binom{a+2t-r-1}{n-1} - \binom{a}{n-1} \right),$$

for $k \geq s+1$

$$C_n(K_t \times P_r,k) = 2t \sum_{i=1}^{s} \binom{a+i-r-i}{n-1} - \binom{a-r+i}{n-1},$$

for $k \geq s+1$.  

125
in which \( \alpha = p-t(k-1)-1 \).

**Proof.** \( C_n(K_t \times P_r, 0) \) and \( C_n(K_t \times P_r, 1) \) are obtained from (1.8) and (1.9). For \( 2 \leq k \leq \delta_n \) we shall consider three cases for the values of \( k \).

1. If \( 2 \leq k < s \), then for each \( 1 \leq i \leq s \) the number of vertices of distance \( k \) from any vertex, say \((u_j, v_i)\), of \( A_i \) is \( t \) and the number of vertices of distance more than \( k \) from \((u_j, v_i)\) is \( p-t(i+k-1)-1 = \alpha - ti \) when \( 1 \leq i \leq k-1 \) which gives us
   \[
   a = \sum_{i=1}^{k-1} \sum_{j=1}^{n} \binom{n-1-j}{j} \left( \alpha - ti \right) \quad \text{...(6.1)}
   \]
   If \( i=k \), then there are \( 2t-1 \) vertices of distance \( k \) from \((u_j, v_i)\), and there are \( p-t(2k-1)-1 \) vertices of distance more than \( k \). This gives us
   \[
   b = \sum_{j=1}^{n-1} \binom{n-1-j}{j} \left( p-2kt+t-1 \right) \quad \text{...(6.2)}
   \]
   If \( k+1 \leq i \leq s \), then there are \( 2t \) vertices of distance \( k \) from \((u_j, v_i)\) and there are \( p-t(2k-1)-2 \) vertices of distance more than \( k \). This gives us
   \[
   c = \sum_{j=1}^{s-k+1} \sum_{j=1}^{n-1} \binom{n-1-j}{j} \left( p-2kt+t-2 \right) \quad \text{...(6.3)}
   \]
   Since \( r=2s \) and each \( A_i \) consists of \( t \) vertices,
   \[
   C_n(K_t \times P_r, k) = 2t(a+b+c) \quad \text{when} \ 2 \leq k < s.
   \]

2. If \( k=s \), then using the same reasoning as in case (1) we find that (6.1) and (6.2) are true for this case, and (6.3) does not hold. Thus,
   \[
   C_n(K_t \times P_r, k) = 2t(a+b) \quad \text{when} \ k=s.
   \]

3. If \( k \geq s+1 \), then it is clear that both (6.2) and (6.3) do not hold. Thus,
   \[
   C_n(K_t \times P_r, k) = 2ta \quad \text{when} \ k \geq s+1.
   \]
   Substituting \( a, b \) and \( c \), we get the required results. \( \blacksquare \)
Theorem 6.3. Let \( r=2s+1, s \geq 1, t \geq 3 \) and \( 3 \leq n \leq rt \). Then

\[
W_n(K_t \times P_r; x) = \sum_{k=0}^{\delta_n} C_n(K_t \times P_r, k)x^k,
\]

where

\[
C_n(K_t \times P_r, 0) = rt \binom{n-1}{n-2},
\]

\[
C_n(K_t \times P_r, 1) = rt \binom{n-1}{n-1} - 2t \binom{n-t-1}{n-1} - t(r-2) \binom{n-r-2}{n-1},
\]

for \( 2 \leq k \leq s \)

\[
C_n(K_t \times P_r, k) = 4t \left( \sum_{i=1}^{k-1} \binom{\alpha+i-n}{n-1} - \binom{\alpha-n}{n-1} \right) + \binom{\alpha+2r-2k-1}{n-1} - \binom{\alpha-k}{n-1},
\]

for \( k = s+1 \),

\[
C_n(K_t \times P_r, k) = 2t \sum_{i=1}^{s} \left\{ \alpha+i-n \choose n-1 \right\} - \alpha-n \choose n-1 \right\} + t \binom{2r-2}{n-1},
\]

for \( s+1 < k \leq \delta_n \),

\[
C_n(K_t \times P_r, k) = 2t \sum_{i=1}^{s} \left\{ \alpha+i-n \choose n-1 \right\} - \alpha-n \choose n-1 \right\},
\]

in which

\[
\alpha = p-t(k-1)-1.
\]

Proof. The proof of \( C_n(K_t \times P_r, k) \) for \( k \neq s+1 \) is similar to that for even \( r \) given in Theorem 6.2. For \( k = s+1 \) we add the number of pairs \( ((u_j, v_{s+1}), S) \) of \( n \)-distance \( s+1 \), which equals \( \binom{2r-2}{n-1} \) for each \( 1 \leq j \leq t \). ■
REFERENCES


